

LINES OF MINIMA AND TEICHMÜLLER GEODESICS

YOUNG-EUN CHOI, KASRA RAFI, AND CAROLINE SERIES

ABSTRACT. For two measured laminations ν^+ and ν^- that fill up a hyperbolizable surface S and for $t \in (-\infty, \infty)$, let \mathcal{L}_t be the unique hyperbolic surface that minimizes the length function $e^t l(\nu^+) + e^{-t} l(\nu^-)$ on Teichmüller space. We characterize the curves that are short in \mathcal{L}_t and estimate their lengths. We find that the short curves coincide with the curves that are short in the surface \mathcal{G}_t on the Teichmüller geodesic whose horizontal and vertical foliations are respectively, $e^t \nu^+$ and $e^{-t} \nu^-$. By deriving additional information about the twists of ν^+ and ν^- around the short curves, we estimate the Teichmüller distance between \mathcal{L}_t and \mathcal{G}_t . We deduce that this distance can be arbitrarily large, but that if S is a once-punctured torus or four-times-punctured sphere, the distance is bounded independently of t .

1. INTRODUCTION

Suppose that ν^+ and ν^- are measured laminations which fill up a hyperbolizable surface S . The object of this paper is to compare two paths in the Teichmüller space $\mathcal{T}(S)$ of S determined by ν^+ and ν^- . The first is the Teichmüller geodesic $\mathcal{G} = \mathcal{G}(\nu^+, \nu^-)$, whose time t Riemann surface $\mathcal{G}_t \in \mathcal{G}$ supports a quadratic differential q_t whose horizontal and vertical foliations are $\nu_t^+ = e^t \nu^+$ and $\nu_t^- = e^{-t} \nu^-$, respectively [8]. The second is the Kerckhoff line of minima $\mathcal{L} = \mathcal{L}(\nu^+, \nu^-)$ [12]. At time t , $\mathcal{L}_t \in \mathcal{L}$ is the unique hyperbolic surface that minimizes the length function $l(\nu_t^+) + l(\nu_t^-) = e^t l(\nu^+) + e^{-t} l(\nu^-)$ on $\mathcal{T}(S)$. (Recall that \mathcal{G} can be characterized as the locus in $\mathcal{T}(S)$ where the product of the extremal lengths $Ext(\nu^+) Ext(\nu^-)$ is minimized [8].) Lines of minima have many properties in common with Teichmüller geodesics, see [12], and have been shown to be closely linked to deforming Fuchsian into quasi-Fuchsian groups by bending, see [25].

We are interested in comparing the two trajectories \mathcal{G} and \mathcal{L} , in particular, to see whether or not they remain a bounded distance apart. If both \mathcal{G}_t and \mathcal{L}_t are contained in the thick part of $\mathcal{T}(S)$, it is relatively easy to show that the Teichmüller distance between them is uniformly bounded independently of t , see Theorem 3.8. A more surprising fact is that in general, the sets of short curves on \mathcal{G} and \mathcal{L} are the same.

Writing $l_{\mathcal{G}_t}(\alpha), l_{\mathcal{L}_t}(\alpha)$ for the geodesic lengths of a simple closed curve α in the hyperbolic metrics on $\mathcal{G}_t, \mathcal{L}_t$ respectively, we prove:

Theorem A (Proposition 7.1 and Corollary 7.9). *The set of short curves on \mathcal{G}_t and \mathcal{L}_t coincide. More precisely, there exist universal constants $\epsilon_1, \dots, \epsilon_4 > 0$ such that for each t , $l_{\mathcal{G}_t}(\alpha) < \epsilon_1$ implies $l_{\mathcal{L}_t}(\alpha) < \epsilon_2$ and $l_{\mathcal{L}_t}(\alpha) < \epsilon_3$ implies $l_{\mathcal{G}_t}(\alpha) < \epsilon_4$.*

Finding combinatorial estimates for the lengths of these short curves occupies the main part of the paper and leads to a coarse estimate of the distance between \mathcal{G}_t and \mathcal{L}_t . It turns out that along both \mathcal{G} and \mathcal{L} there are two distinct reasons why a curve α can become short: either the relative twisting of ν^+ and ν^- about α is large, or ν^+ and ν^- have large relative complexity in $S \setminus \alpha$ (the completion of the surface S minus α), in the sense that every essential arc or closed curve in $S \setminus \alpha$ must have large intersection with ν^+ or ν^- . Our results give sufficient control to construct examples which show that \mathcal{G}_t and \mathcal{L}_t may or may not remain a bounded distance apart.

The estimates for curves which become short along \mathcal{G} are based on Rafi [21]. For convenience we say a curve is ‘extremely short’ on a given surface if its hyperbolic length is less than some fixed constant $\epsilon_0 > 0$ defined in terms of the Margulis constant, see Section 2.1. Rafi’s results imply:

Theorem B (Theorem 5.10). *Suppose that α is extremely short on \mathcal{G}_t . Then*

$$\frac{1}{l_{\mathcal{G}_t}(\alpha)} \asymp \max\{D_t(\alpha), \log K_t(\alpha)\}.$$

The terms $D_t(\alpha)$ and $K_t(\alpha)$ correspond respectively to the relative twisting and large relative complexity mentioned above. More precisely,

$$D_t(\alpha) = e^{-2|t-t_\alpha|} d_\alpha(\nu^+, \nu^-)$$

where t_α is the *balance time* at which $i(\alpha, \nu_t^+) = i(\alpha, \nu_t^-)$ and $d_\alpha(\nu^+, \nu^-)$ is the relative twisting, that is, the difference between the twisting of ν^+ and ν^- around α , see Section 4.3. The term $K_t(\alpha)$ depends on the (possibly coincident) thick components Y_1, Y_2 that are adjacent to α in the thick-thin decomposition of \mathcal{G}_t . Let q_t be the area 1 quadratic differential on \mathcal{G}_t whose horizontal and vertical foliations are respectively ν_t^+ and ν_t^- . Associated to q_t is a singular Euclidean metric; we denote the geodesic length of a curve γ in this metric by $l_{q_t}(\gamma)$, see Section 2.6. By definition

$$K_t(\alpha) = \max\left\{\frac{\lambda_{Y_1}}{l_{q_t}(\alpha)}, \frac{\lambda_{Y_2}}{l_{q_t}(\alpha)}\right\}$$

where λ_{Y_i} is the length of the shortest non-trivial, non-peripheral simple closed curve on Y_i with respect to the q_t -metric, see Section 5.3.

One of the main results of this paper is a similar characterization of curves which become short along $\mathcal{L}(\nu^+, \nu^-)$. We prove that the hyperbolic length $l_{\mathcal{L}_t}(\alpha)$ of a short curve in \mathcal{L}_t is estimated as follows:

Theorem C (Theorem 7.13). *Suppose that α is extremely short on \mathcal{L}_t . Then*

$$\frac{1}{l_{\mathcal{L}_t}(\alpha)} \asymp \max\{D_t(\alpha), \sqrt{K_t(\alpha)}\}.$$

The main tool in the proof is the well-known derivative formula of Kerckhoff [11] and Wolpert [27] for the variation of length with respect to Fenchel-Nielsen twist, together with the extension proved by Series [24] for variation with respect to the lengths of pants curves.

To estimate the Teichmüller distance between two surfaces that have the same set of short curves one uses Minsky's product region theorem [20]. To apply this, in addition to Theorems B and C, we need to estimate the Teichmüller distance between the hyperbolic thick components of \mathcal{G}_t and \mathcal{L}_t , and also the difference between the Fenchel-Nielsen twist coordinates corresponding to the short curves in the two surfaces. In Theorem 7.10 and Corollary 7.11, we show that the Teichmüller distance between the corresponding thick components is bounded. In Theorem 6.2, we estimate the twist of ν^+ and ν^- around α at \mathcal{L}_t . Combined with the analogous estimate for \mathcal{G}_t proved in [22], we are able to show that the contribution to the Teichmüller distance between \mathcal{G}_t and \mathcal{L}_t from the twisting is dominated by that from the lengths, leading to

Theorem D (Theorem 7.15). *The Teichmüller distance between \mathcal{G}_t and \mathcal{L}_t is given by*

$$d_{\mathcal{T}(S)}(\mathcal{G}_t, \mathcal{L}_t) = \frac{1}{2} \log \max \left\{ \frac{l_{\mathcal{G}_t}(\alpha)}{l_{\mathcal{L}_t}(\alpha)} \right\} \pm O(1),$$

where the maximum is taken over all curves α that are short in \mathcal{G}_t .

Theorems B, C, and D enable us to construct the various examples alluded to above. Because $K_t(\alpha)$ can become arbitrarily large while $D_t(\alpha)$ remains bounded, it follows that \mathcal{G}_t and \mathcal{L}_t do not always remain a bounded distance apart. However, in the case in which S is a once-punctured torus or four-times-punctured sphere, it turns out that the quantity $K_t(\alpha)$ is always bounded and therefore that the two paths are always within bounded distance of each other. These ideas are taken further in [5], where we show that \mathcal{L}_t is a Teichmüller quasi-geodesic.

The greater part of the work of this paper is contained in the proof of Theorem C. It is carried out in several steps. First, using the derivative formulae mentioned above, we show in Theorem 6.1 that the length may be estimated by a formula identical to that in Theorem C, except that $K_t(\alpha)$ is replaced by another geometric quantity

$$H_t(\alpha) = \sup_{\beta \in \mathcal{B}} \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)}.$$

Here \mathcal{B} are those pants curves in a short pants decomposition of \mathcal{L}_t (see Section 3.1) which are boundaries of pants adjacent to α , while as above l_{q_t} denotes length in the singular Euclidean metric associated to q_t . This is the content of Section 6.

We now need to compare $H_t(\alpha)$ and $K_t(\alpha)$. From the definition it is quite easy to show (Proposition 7.1) that $H_t(\alpha) \succ K_t(\alpha)$. In particular, it follows that a curve that is short in \mathcal{G}_t is at least as short in \mathcal{L}_t . Next, we show in Proposition 7.8 that on a subsurface whose injectivity radius is bounded below in \mathcal{G}_t , the injectivity radius with respect to \mathcal{L}_t is also bounded below, perhaps by a smaller constant. The main point in the proof is Proposition 7.4, which shows that the hyperbolic metric on \mathcal{L}_t not only minimizes the sum of lengths $l(\nu_t^+) + l(\nu_t^-)$, but also, up to multiplicative error, it minimizes the contribution of $l(\nu_t^+) + l(\nu_t^-)$ to each thick component of the thick-thin decomposition of \mathcal{G}_t . This proves Theorem A.

Having set up a one-to-one correspondence between the thick components of \mathcal{G}_t and \mathcal{L}_t , we show in Theorem 7.10 and Corollary 7.11 that the Teichmüller distance between corresponding thick components is bounded. Finally we are able to prove in Proposition 7.12 that $H_t(\alpha) \prec K_t(\alpha)$, completing the proof of Theorem C.

Prior to this paper, the only results related to the relative behavior of \mathcal{G} and \mathcal{L} were some partial results about their behavior at infinity. Results of Masur [15] (for Teichmüller geodesics) and of Díaz and Series [6] (for lines of minima) show that if either ν^\pm are supported on closed curves, or if ν^\pm are uniquely ergodic, then \mathcal{G} and \mathcal{L} limit on the same points in the Thurston boundary of $\mathcal{T}(S)$. In general, the question of the behavior at infinity remains unresolved, but see also [13] which shows that there are Teichmüller geodesics \mathcal{G} which do not converge in Thurston's compactification of $\mathcal{T}(S)$. It is not hard to apply the results of this paper to show the same is true of lines of minima in Lenzhen's example; we hope to explore this in more detail elsewhere.

The motivation for our approach stems in part from a central ingredient of the proof of the ending lamination theorem [4]. Suppose

that N is a hyperbolic 3-manifold homeomorphic to $S \times \mathbb{R}$. The ending lamination theorem states that N is completely determined by the asymptotic invariants of its two ends. A key step is to show that if these end invariants are induced by the laminations ν^+ and ν^- , then the curves on S which have short geodesic representatives in N can be characterized in terms of their combinatorial relationship to ν^+ and ν^- . (The relationship is expressed using the complex of curves of S , details of which are not needed in what follows. Roughly speaking, a curve is short in N if and only if the distance between the projections of ν^+ and ν^- to some subsurface $Y \subset S$ is large in the curve complex of Y .) In [21], Rafi found a similar combinatorial characterization which shows that the curves which are short in N are almost, but not quite, the same as those curves which become short along $\mathcal{G}(\nu^+, \nu^-)$. Our definition of K_t is closely related to Rafi's study [23] of the relationship between the thick-thin decomposition of a hyperbolic surface S and a quadratic differential metric on the same surface. The relationship between these two metrics plays a key role throughout the paper.

The paper is organized as follows. In Section 2, we recall some background facts about lines of minima and Teichmüller geodesics. In Section 3, we prove Theorem 3.8 mentioned above, which states that if both \mathcal{G}_t and \mathcal{L}_t are contained in the thick part of $\mathcal{T}(S)$, then the Teichmüller distance between them is bounded. We hope that treating this special case separately early on will give some intuition about what needs to be done in general. In Section 4, we review twists and Fenchel-Nielsen coordinates and in Section 5, after reviewing some fundamental facts about quadratic differential metrics, we derive Theorem B and state the estimates for twists about short curves proved in [22]. In Section 6, we prove Theorem 6.1 and derive estimates for twists about the short curves. Finally, in Section 7, we prove Theorems A, C and D. Throughout the paper, we make use of several basic length estimates on hyperbolic surfaces. The proofs, being somewhat long but relatively straightforward, are relegated to the Appendix.

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2. PRELIMINARIES

Throughout, S is an orientable hyperbolizable surface of finite type, possibly with punctures but with no other boundary.

2.1. Thick-thin decomposition. Let \mathcal{S} denote the set of free homotopy classes of non-peripheral, non-trivial simple closed curves on S . If (S, σ) is a surface with hyperbolic metric σ and $\alpha \in \mathcal{S}$, we write $l_\sigma(\alpha)$ for the hyperbolic length of the unique geodesic representative of α with respect to σ . The Margulis lemma provides a universal constant $\epsilon_{\mathcal{M}} > 0$ such that all components of the $\epsilon_{\mathcal{M}}$ -thin part of (S, σ) (i.e. the subset of S where the injectivity radius is less than $\epsilon_{\mathcal{M}}$) are horocyclic neighborhoods of cusps or annular collars about short geodesics. The $\epsilon_{\mathcal{M}}$ -thick part of the surface is the complement of the thin part.

For our purposes, it is necessary to choose a constant $\epsilon_0 > 0$ sufficiently smaller than $\epsilon_{\mathcal{M}}$, in order that the ϵ_0 -thick-thin decomposition of a surface satisfies certain geometric conditions. These conditions will be mentioned when the context arises, but we assume that ϵ_0 has been chosen once and for all so that these conditions are met. If $l_\sigma(\alpha) < \epsilon_0$, we shall say that α is *extremely short* in σ .

2.2. Notation. Since we will be dealing mainly with coarse estimates, we want to avoid keeping track of constants which are universal, in that they do not depend on any specific metric or curve under discussion. For functions f, g we write $f \prec g$ and $f \stackrel{*}{\prec} g$ to mean respectively, that there are constants $c > 1, C > 0$, depending only on the topology of S and the fixed constant ϵ_0 , such that

$$\frac{1}{c}g(x) - C \leq f(x) \leq cg(x) + C \quad \text{and} \quad \frac{1}{c}g(x) \leq f(x) \leq cg(x).$$

The symbols $\prec, \stackrel{*}{\prec}, \succ, \stackrel{*}{\succ}$ are defined similarly. For a positive quantity X , we often write $X = O(1)$ instead of $X \prec 1$ to indicate X is bounded above by a constant depending only on the topology of S and ϵ_0 , and more generally we write $X = O(Y)$ to mean that $X/Y = O(1)$ for a positive function Y .

2.3. Measured laminations. We denote the space of measured laminations on S by $\mathcal{ML}(S)$. Given any hyperbolic metric σ on S , a measured lamination $\xi \in \mathcal{ML}(S)$ can be realized as a geodesic measured lamination with respect to σ . The hyperbolic length function extends by linearity and continuity to $\mathcal{ML}(S)$; we write $l_\sigma(\xi)$ for the hyperbolic length of a lamination $\xi \in \mathcal{ML}(S)$. The geometric intersection number $i(\alpha, \beta)$ of curves $\alpha, \beta \in \mathcal{S}$ also extends continuously to $\mathcal{ML}(S)$. Laminations $\mu, \nu \in \mathcal{ML}(S)$ are said to *fill up* S if $i(\mu, \xi) + i(\nu, \xi) > 0$ for all $\xi \in \mathcal{ML}(S)$. For $\xi \in \mathcal{ML}(S)$, we denote the underlying leaves by $|\xi|$.

2.4. Teichmüller space. The Teichmüller space $\mathcal{T}(S)$ of S is the space of all conformal structures on S up to isotopy. The Teichmüller distance $d_{\mathcal{T}(S)}(\Sigma, \Sigma')$ between two marked Riemann surfaces $\Sigma, \Sigma' \in \mathcal{T}(S)$ is $[\log K]/2$, where K is the smallest quasiconformal constant of a homeomorphism from Σ to Σ' which is isotopic to the identity.

Each conformal structure $\Sigma \in \mathcal{T}(S)$ is uniformized by a unique hyperbolic structure σ , and conversely, each hyperbolic structure σ has an underlying conformal structure Σ . Thus, we also consider $\mathcal{T}(S)$ to be the space of all hyperbolic metrics on S up to isotopy. The *thick part* $\mathcal{T}_{thick}(S)$ of $\mathcal{T}(S)$ will be defined as the subset of all hyperbolic metrics such that every closed geodesic has length bounded below by the constant ϵ_0 .

2.5. Kerckhoff lines of minima. Suppose that $\nu^+, \nu^- \in \mathcal{ML}(S)$ fill up S . Kerckhoff [12] showed that the length function

$$\sigma \mapsto l_\sigma(\nu^+) + l_\sigma(\nu^-)$$

has a global minimum on $\mathcal{T}(S)$ at a unique surface \mathcal{L}_0 . Moreover, as t varies in $(-\infty, \infty)$, the surface $\mathcal{L}_t \in \mathcal{T}(S)$ that realizes the global minimum of $l(\nu_t^+) + l(\nu_t^-)$ for the weighted laminations $\nu_t^+ = e^t \nu^+$ and $\nu_t^- = e^{-t} \nu^-$ varies continuously with t and traces out a path $t \mapsto \mathcal{L}_t$ called the *line of minima* $\mathcal{L}(\nu^+, \nu^-)$ of ν^\pm .

2.6. Quadratic differentials. We give a brief summary of facts about quadratic differentials that we use and refer the reader to [7, 26] for a detailed and comprehensive background. Let Σ be a Riemann surface and q a quadratic differential on Σ which is holomorphic, except possibly at punctures, where q may have a pole of order one. This ensures that the area of Σ with respect to the area element $|q(z)dz^2|$ is finite, and we normalize so that the area is 1. Let $\mathcal{Q}(\Sigma)$ be the space of all such meromorphic quadratic differentials on Σ .

The zeros and poles of q are called *critical points*. Away from the critical points, we have two mutually orthogonal line fields defined respectively by the conditions that $\text{Im}[\sqrt{q(z)}dz]$ is zero and $\text{Re}[\sqrt{q(z)}dz]$ is zero. This defines a pair of measured singular foliations on Σ with singularities at the critical points of q , respectively called the *horizontal foliation* \mathcal{H}_q and the *vertical foliation* \mathcal{V}_q . The measures on these foliations are determined by integrating the line element $|\sqrt{q(z)}dz|$. More precisely, for a curve η , its horizontal and vertical measures are given respectively by

$$h_q(\eta) = \int_\eta |\text{Re}[\sqrt{q(z)}dz|, \quad v_q(\eta) = \int_\eta |\text{Im}[\sqrt{q(z)}dz|.$$

We call $h_q(\eta)$ and $v_q(\eta)$ respectively, the *horizontal length* and the *vertical length* of η .

Every essential simple closed curve γ in (S, q) has a unique q -geodesic representative, unless it is in a family of closed Euclidean geodesics foliating an annulus whose interior contains no singularities. We denote the q -geodesic length of γ by $l_q(\gamma)$. It satisfies the following inequalities:

$$(1) \quad [h_q(\gamma) + v_q(\gamma)]/\sqrt{2} \leq l_q(\gamma) \leq h_q(\gamma) + v_q(\gamma).$$

By definition of intersection numbers for measured foliations, we have $v_q(\gamma) = i(\mathcal{H}_q, \gamma)$ and $h_q(\gamma) = i(\mathcal{V}_q, \gamma)$, so Equation (1) implies

$$(2) \quad l_q(\gamma) \stackrel{*}{\asymp} i(\mathcal{V}_q, \gamma) + i(\mathcal{H}_q, \gamma).$$

This approximation will be used repeatedly.

2.7. Teichmüller geodesics. Suppose that $\Sigma, \Sigma' \in \mathcal{T}(S)$ are marked Riemann surfaces with $d_{\mathcal{T}(S)}(\Sigma, \Sigma') = d$. Then there is a unique quadratic differential q on Σ such that the conformal structure on Σ' is obtained from that of Σ by expanding in the horizontal direction of q by a factor e^d and contracting in the vertical direction by e^{-d} . The homeomorphism which realizes this is called the Teichmüller map from Σ to Σ' , and has quasiconformal distortion e^{2d} . The 1-parameter family of quadratic differentials q_t whose horizontal and vertical foliations are respectively, $e^t \mathcal{H}_q$ and $e^{-t} \mathcal{V}_q$, for $0 \leq t \leq d$, define the geodesic path from Σ to Σ' with respect to the Teichmüller metric.

Gardiner and Masur [8] showed that for any pair of measured laminations $\nu^+, \nu^- \in \mathcal{ML}(S)$ which fill up S and such that $i(\nu^+, \nu^-) = 1$, there is a unique Riemann surface $\Sigma \in \mathcal{T}(S)$ and a unique quadratic differential $q \in \mathcal{Q}(\Sigma)$ whose horizontal and vertical foliations are ν^+, ν^- respectively. (This uses the one-to-one correspondence between the space of measured laminations and the space of measured foliations.) For $t \in \mathbb{R}$ set

$$\nu_t^+ = e^t \nu^+, \quad \nu_t^- = e^{-t} \nu^-$$

and let \mathcal{G}_t and q_t be the corresponding Riemann surface and quadratic differential. The path $t \mapsto \mathcal{G}_t$ defines a Teichmüller geodesic which we denote $\mathcal{G} = \mathcal{G}(\nu^+, \nu^-)$. We abuse notation and use \mathcal{G}_t to also denote the hyperbolic metric that uniformizes the Riemann surface \mathcal{G}_t .

2.8. The balance time. Let $\alpha \in \mathcal{S}$. We say α is *vertical* along $\mathcal{G}(\nu^+, \nu^-)$ if its intersection $i(\alpha, \nu^-)$ with the vertical foliation ν^- vanishes. In this case, α can be realized as a union of leaves of the vertical foliation. Similarly, α is *horizontal* if $i(\alpha, \nu^+) = 0$. Mostly we shall be dealing with curves α which are neither horizontal nor vertical. In this case, there is always a unique time t_α at which $i(\alpha, \nu_{t_\alpha}^+) = i(\alpha, \nu_{t_\alpha}^-)$. We

call t_α the *balance time* of α . The length of α with respect to \mathcal{G}_t is approximately convex along \mathcal{G} and is close to its minimum at t_α , see [22] Theorem 3.1. Our estimation of the hyperbolic lengths of short curves will mainly be made relative to their balance time.

3. COMPARISON ON THE THICK PART OF TEICHMÜLLER SPACE

In this section we prove Theorem 3.8, which states that if \mathcal{G}_t and \mathcal{L}_t are in the ϵ_0 -thick part $\mathcal{T}_{thick}(S)$ of Teichmüller space, then the Teichmüller distance between them is uniformly bounded by a constant that depends only on the topology of S and ϵ_0 . The idea is to first approximate the length of a curve ζ for any $\sigma \in \mathcal{T}_{thick}(S)$ by its intersection with what we call a *short marking* for σ , and then to compare the short markings for \mathcal{G}_t and \mathcal{L}_t . This method will be extended in Section 7 when we consider the Teichmüller distance between \mathcal{G}_t and \mathcal{L}_t in general. We begin with some definitions.

3.1. Short markings. We call a maximal collection of pairwise disjoint, homotopically distinct, non-peripheral, non-trivial simple closed curves on S , a *pants curve system* on S . The terminology is due to the fact that the complementary components are pairs of pants, i.e., three holed spheres (in which some boundary components may be punctures). Our notion of a *marking* is motivated by [17]:

Definition 3.1. A marking M on a surface S is a system of pants curves $\alpha_1, \dots, \alpha_k$ and simple closed curves $\delta_{\alpha_1}, \dots, \delta_{\alpha_k}$ such that

$$\begin{cases} i(\alpha_i, \delta_{\alpha_j}) = 0 & \text{if } i \neq j \\ i(\alpha_i, \delta_{\alpha_i}) = 2 & \text{if two distinct pairs of pants are adjacent along } \alpha_i \\ i(\alpha_i, \delta_{\alpha_i}) = 1 & \text{if } \alpha_i \text{ is adjacent to only a single pair of pants.} \end{cases}$$

We call δ_{α_i} the *dual curve* of α_i .

In the second case, $\alpha_i \cup \delta_{\alpha_i}$ fill a four-holed sphere (that is, a regular neighborhood of $\alpha_i \cup \delta_{\alpha_i}$ is homeomorphic to a four-holed sphere) and in the third case $\alpha_i \cup \delta_{\alpha_i}$ fill a one-holed torus. It is easy to see that any two markings which have the same pants system \mathcal{P} have dual curves which differ only by twists and half-twists around the curves in \mathcal{P} .

The following well-known lemma states that for any hyperbolic metric, one can always choose a pants system whose length is universally bounded:

Lemma 3.2. (Bers [3]) *There exists a constant $L > 0$ such that for every $\sigma \in \mathcal{T}(S)$ there is a pants curve system \mathcal{P} with the property that $l_\sigma(\alpha) < L$ for every $\alpha \in \mathcal{P}$.*

If the boundary curves of a pair of pants have bounded length as in Bers's lemma, the geometry of a pair of pants satisfies the following (for a proof, see Appendix):

Lemma 3.3. *Let P be a totally geodesic pair of pants with boundary curves $\alpha_1, \alpha_2, \alpha_3$ of lengths $l(\alpha_i) < L$ for $i = 1, 2, 3$. Then the common perpendicular of α_i, α_j (where possibly $i = j$) has length*

$$\log \frac{1}{l(\alpha_i)} + \log \frac{1}{l(\alpha_j)} \pm O(1),$$

where the bound on the error depends only on L .

We will say that a pants curve system as in Lemma 3.2 is *short* in (S, σ) . A *short marking* M_σ for σ is a short pants system together with a dual system chosen so that each dual curve δ_{α_i} is the shortest among all possible dual curves. For a given pants curve, notice that there may be more than one shortest dual curve, in which case any choice will suffice. Also notice that *not all curves in a short marking are necessarily short*; if a pants curve is very short, then the corresponding dual curve will be very long. More precisely, we have the following easy consequence of Lemma 3.3:

Corollary 3.4. *Let M_σ be a short marking for σ and let $\alpha, \delta_\alpha \in M_\sigma$ be a pants curve and its dual. Then*

$$l_\sigma(\delta_\alpha) = i(\delta_\alpha, \alpha) \cdot 2 \log \frac{1}{l_\sigma(\alpha)} \pm O(1).$$

Observe that if $\sigma \in \mathcal{T}_{thick}(S)$, since the length of every curve in M_σ is uniformly bounded below, it follows from Lemma 3.2 and Corollary 3.4 that the length of every curve in M_σ is also uniformly bounded above. Thus, if $\sigma \in \mathcal{T}_{thick}(S)$, we have $l_\sigma(M_\sigma) \stackrel{*}{\asymp} 1$.

For surfaces in the thick part of $\mathcal{T}(S)$, short markings coarsely determine the geometry. We express this in the following proposition whose proof can be found in the proof of Lemma 4.7 in [19] (see also the proof of Proposition 7.7 below). If M is a marking and $\xi \in \mathcal{ML}(S)$, we write

$$i(M, \xi) = \sum_{\gamma \in M} i(\gamma, \xi).$$

Proposition 3.5. *Let M_σ be a short marking for $\sigma \in \mathcal{T}_{thick}(S)$. Then for any $\zeta \in \mathcal{S}$,*

$$l_\sigma(\zeta) \stackrel{*}{\asymp} i(M_\sigma, \zeta).$$

Since both length and intersection number scale linearly with weights of simple closed curves, it follows that:

Proposition 3.6. *Let M_σ be a short marking for $\sigma \in \mathcal{T}_{thick}(S)$. Then for any $\xi \in \mathcal{ML}(S)$,*

$$l_\sigma(\xi) \stackrel{*}{\asymp} i(M_\sigma, \xi).$$

3.2. Comparison on the thick part. We use the estimate in Proposition 3.6 to compare \mathcal{G}_t and \mathcal{L}_t in the thick part of $\mathcal{T}(S)$. The following well-known lemma is proved in greater generality in [23] (see Theorem 5.5(ii) below). Recall that Σ denotes the conformal structure associated to the metric σ .

Lemma 3.7. *Suppose that $\sigma \in \mathcal{T}_{thick}(S)$ and $q \in \mathcal{Q}(\Sigma)$. Then for every $\zeta \in \mathcal{S}$,*

$$l_\sigma(\zeta) \stackrel{*}{\asymp} l_q(\zeta).$$

Theorem 3.8. *If $\mathcal{G}_t, \mathcal{L}_t \in \mathcal{T}_{thick}(S)$ then $d_{\mathcal{T}(S)}(\mathcal{G}_t, \mathcal{L}_t) = O(1)$.*

Proof. By Lemma 3.7, Equation (2), and Proposition 3.6, we have

$$(3) \quad \begin{aligned} l_{\mathcal{G}_t}(M_{\mathcal{G}_t}) &\stackrel{*}{\asymp} l_{q_t}(M_{\mathcal{G}_t}) \stackrel{*}{\asymp} i(M_{\mathcal{G}_t}, \nu_t^+) + i(M_{\mathcal{G}_t}, \nu_t^-) \\ &\stackrel{*}{\asymp} l_{\mathcal{G}_t}(\nu_t^+) + l_{\mathcal{G}_t}(\nu_t^-). \end{aligned}$$

Now, since \mathcal{L}_t minimizes $l_\sigma(\nu_t^+) + l_\sigma(\nu_t^-)$ over all $\sigma \in \mathcal{T}(S)$, we have

$$l_{\mathcal{G}_t}(\nu_t^+) + l_{\mathcal{G}_t}(\nu_t^-) \geq l_{\mathcal{L}_t}(\nu_t^+) + l_{\mathcal{L}_t}(\nu_t^-).$$

Reversing the sequence of estimates in Equation (3), we get

$$\begin{aligned} l_{\mathcal{L}_t}(\nu_t^+) + l_{\mathcal{L}_t}(\nu_t^-) &\stackrel{*}{\asymp} i(M_{\mathcal{L}_t}, \nu_t^+) + i(M_{\mathcal{L}_t}, \nu_t^-) \stackrel{*}{\asymp} l_{q_t}(M_{\mathcal{L}_t}) \\ &\stackrel{*}{\asymp} l_{\mathcal{G}_t}(M_{\mathcal{L}_t}). \end{aligned}$$

Putting together the preceding three equations, we have

$$l_{\mathcal{G}_t}(M_{\mathcal{G}_t}) \stackrel{*}{\asymp} l_{\mathcal{G}_t}(M_{\mathcal{L}_t}).$$

Since $\mathcal{G}_t \in \mathcal{T}_{thick}(S)$, it follows from the observation following Corollary 3.4 that

$$(4) \quad l_{\mathcal{G}_t}(M_{\mathcal{L}_t}) \stackrel{*}{\prec} 1.$$

Notice also that $l_{\mathcal{L}_t}(M_{\mathcal{L}_t}) \stackrel{*}{\prec} 1$. Lemma 4.7 of [19] implies that for any given $B > 0$, the diameter of the set $\{\sigma \in \mathcal{T}(S) : l_\sigma(M_{\mathcal{L}_t}) < B\}$, with respect to the Teichmüller distance, is bounded above by a constant that depends only on B . Thus it follows that $d_{\mathcal{T}(S)}(\mathcal{G}_t, \mathcal{L}_t) = O(1)$. \square

4. TWISTS AND FENCHEL-NIELSEN COORDINATES

In order to compare surfaces in the thin part of Teichmüller space, our main tool will be Minsky's product region theorem [20]. This uses Fenchel-Nielsen coordinates to give a nice coarse expression for Teichmüller distance between surfaces which have common thin parts. To state the results precisely, we first discuss twists and Fenchel-Nielsen coordinates.

4.1. Twists in hyperbolic metrics. There are various ways to define the twist of one curve around another, all of which differ by factors unimportant to us here. We shall follow Minsky [20]. Let $\sigma \in \mathcal{T}(S)$ be a hyperbolic metric and let α be an oriented simple closed geodesic on (S, σ) . Let ζ be a simple geodesic that intersects α transversely and let p be a point of intersection. In the universal cover \mathbb{H}^2 , a lift $\tilde{\zeta}$ of ζ intersects a lift $\tilde{\alpha}$ of α at a lift \tilde{p} of p , and has endpoints ζ_R, ζ_L on $\partial_\infty \mathbb{H}^2$ to the right and left of $\tilde{\alpha}$, respectively (see Figure 1). Let p_R, p_L be the

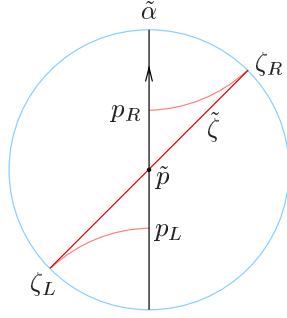


FIGURE 1. Defining the twist of ζ around α .

orthogonal projections of ζ_R, ζ_L to $\tilde{\alpha}$ respectively. Then the twist of ζ around α at p is defined as

$$tw_\sigma(\zeta, \alpha, p) = \pm \frac{d_{\mathbb{H}^2}(p_R, p_L)}{l_\sigma(\alpha)},$$

where the sign is $(+)$ if the direction from p_L to p_R coincides with the orientation of $\tilde{\alpha}$ and $(-)$ if it is opposite. For any other point $q \in \zeta \cap \alpha$, the twist satisfies ([20, Lemma 3.1])

$$|tw_\sigma(\zeta, \alpha, q) - tw_\sigma(\zeta, \alpha, p)| \leq 1.$$

To obtain a number that is independent of the point of intersection, Minsky defines

$$tw_\sigma(\zeta, \alpha) = \min_{p \in \zeta \cap \alpha} tw_\sigma(\zeta, \alpha, p).$$

For convenience, we write $Tw_\sigma(\zeta, \alpha)$ for $|tw_\sigma(\zeta, \alpha)|$.

Note that the definition of twist is valid even if the simple geodesic ζ is not closed, because the inequality

$$|tw_\sigma(\zeta, \alpha, q) - tw_\sigma(\zeta, \alpha, p)| \leq 1$$

depends only on the fact that different lifts of ζ are disjoint. Thus if ν is a measured geodesic lamination that intersects α transversely, we can define the twist of ν around α by taking the infimum of twists over all leaves of ν that intersect α . We remark that although we will be working with measured geodesic laminations, when defining the twist, the measure is irrelevant, in other words the twist $tw_\sigma(\nu, \alpha)$ depends only on the underlying lamination $|\nu|$.

4.2. Fenchel-Nielsen coordinates. We define the Fenchel-Nielsen coordinates

$$(l_\sigma(\alpha_i), s_{\alpha_i}(\sigma))_{i=1}^k$$

associated to a pants curves system $\alpha_1, \dots, \alpha_k$ in the following standard way, see for example [20]. Suppose that P is a pair of pants that is the closure of a component of $S \setminus \{\alpha_1, \dots, \alpha_k\}$. By a *seam* of P , we mean a common perpendicular between two distinct boundary components of P . (Notice that the definition of seam refers to the internal geometry of P alone; two distinct boundary curves of P may project to the same curve on S .) Each boundary curve of P is bisected by the two points at which it meets the two seams intersecting it. We first construct a base surface $\sigma_0 = \sigma_0(l_1^0, \dots, l_k^0)$ in which the pants curve α_i has some specific choice of length l_i^0 . Each α_i is adjacent to two (possibly coincident) pairs of pants; we glue these two pants together in such a way that seams incident on α_i from the two sides match up. Since the seams meet the pants curves orthogonally, they glue up to form a collection of closed geodesics γ_j . Any other structure $\sigma \in \mathcal{T}(S)$ comes endowed with an associated homeomorphism $h : \sigma_0 \rightarrow \sigma$. The length coordinates of σ are defined by $l_\sigma(\alpha_i)$. Let $\sigma_0(l_1, \dots, l_k)$ denote the surface in which α_i (more precisely $h(\alpha_i)$) has length $l_i = l_\sigma(\alpha_i)$, while the curves formed by gluing the new seams are exactly the images $h(\gamma_j)$. Now define $\tau_{\alpha_i}(\sigma)$ to be the signed distance that one has to twist around α_i to obtain σ starting from $\sigma_0(l_1, \dots, l_k)$, where the sign is determined relative to a fixed orientation on σ_0 and hence on σ . Finally we define the twist coordinates of σ by

$$s_{\alpha_i}(\sigma) = \frac{\tau_{\alpha_i}(\sigma)}{l_\sigma(\alpha_i)} \in \mathbb{R}.$$

Lemma 4.1. (*Minsky [20] Lemma 3.5*) *For any lamination $\nu \in \mathcal{ML}(S)$ that intersects $\alpha = \alpha_i$ and any two metrics $\sigma, \sigma' \in \mathcal{T}(S)$,*

$$|(tw_\sigma(\nu, \alpha) - tw_{\sigma'}(\nu, \alpha)) - (s_\alpha(\sigma) - s_\alpha(\sigma'))| \leq 4.$$

In [20], the statement is only given for closed curves. However the argument extends without change to laminations. This is because the proof in [20] is based on the observation that for any two simple closed curves ζ_1, ζ_2 intersecting α , the difference $tw_\sigma(\zeta_1, \alpha) - tw_\sigma(\zeta_2, \alpha)$ is a topological quantity, independent of σ , up to a bounded error of 1. More precisely, it follows from the proof in [20] that if \tilde{S} is the annular cover of S corresponding to α , and if $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ are respectively, lifts of ζ_1 and ζ_2 intersecting the core $\tilde{\alpha}$ of \tilde{S} , then the difference $tw_\sigma(\zeta_1, \alpha) - tw_\sigma(\zeta_2, \alpha)$ is the signed intersection of $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ in the annulus \tilde{S} , up to a bounded error. This topological characterization holds even when ζ_1 and ζ_2 are simple geodesics which are not necessarily closed.

4.3. Relative twist in an annulus. The above topological observation allows us to define the following:

Definition 4.2. *For any two laminations ν_1, ν_2 that intersect a curve α , define their algebraic intersection around α to be*

$$i_\alpha(\nu_1, \nu_2) = \inf_{\sigma} [tw_\sigma(\nu_1, \alpha) - tw_\sigma(\nu_2, \alpha)],$$

where the infimum is taken over all possible surfaces $\sigma \in \mathcal{T}(S)$.

Often we need only the absolute value:

Definition 4.3. *For any two laminations ν_1, ν_2 that intersect a curve α , define the relative twisting of ν_1, ν_2 around α to be*

$$d_\alpha(\nu_1, \nu_2) = |i_\alpha(\nu_1, \nu_2)|.$$

Thus for any $\sigma \in \mathcal{T}(S)$, we have

$$|tw_\sigma(\nu_1, \alpha) - tw_\sigma(\nu_2, \alpha)| = d_\alpha(\nu_1, \nu_2) + O(1).$$

Notice that $i_\alpha(\nu_1, \nu_2)$ and $d_\alpha(\nu_1, \nu_2)$ are independent of the measures on ν_1, ν_2 , depending only on the underlying laminations $|\nu_1|$ and $|\nu_2|$.

Using Definition 4.2, one sees easily that

$$(5) \quad i_\alpha(\nu_1, \nu_2) = i_\alpha(\nu_1, \xi) - i_\alpha(\nu_2, \xi) + O(1)$$

for any curve ξ transverse to α . It is also easily seen that $d_\alpha(\nu_1, \nu_2)$ agrees up to $O(1)$ with the definition of subsurface distance between the projections of $|\nu_1|$ and $|\nu_2|$ to the annular cover of S with core α , as defined in [17] Section 2.4 and used throughout [21, 22].

Another essentially equivalent way of measuring twist is to look at the intersection with the shortest curve transverse to α :

Lemma 4.4. *Let α be a pants curve and let δ_α be a shortest dual curve of α in some marking for σ . Then for any simple closed curve ζ intersecting α , we have $|tw_\sigma(\zeta, \alpha) - i_\alpha(\zeta, \delta_\alpha)| = O(1)$.*

Proof. Since $i_\alpha(\zeta, \delta_\alpha) = tw_\sigma(\zeta, \alpha) - tw_\sigma(\delta_\alpha, \alpha)$, up to a bounded error of 1, it is sufficient to show that $|tw_\sigma(\delta_\alpha, \alpha)| = O(1)$.

Let $\tilde{\delta}_\alpha$ be a lift of δ_α in the universal cover \mathbb{H}^2 and let $\tilde{\alpha}, \tilde{\alpha}'$ be the two lifts of α containing the endpoints of $\tilde{\delta}_\alpha$ (see Figure 2). Let η be the perpendicular between $\tilde{\alpha}$ and $\tilde{\alpha}'$ and let p, p' be the endpoints of η

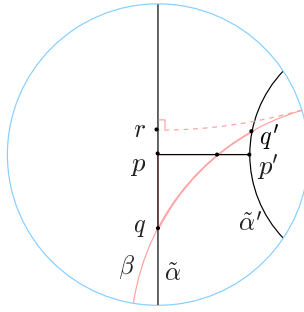


FIGURE 2. Bound on the twist of δ_α around α .

on $\tilde{\alpha}, \tilde{\alpha}'$, respectively. Since δ_α is the shortest dual curve, the endpoints of $\tilde{\delta}_\alpha$ must be within distance $l_\sigma(\alpha)$ from p, p' . Let q, q' be points on $\tilde{\alpha}, \tilde{\alpha}'$ at distance $l_\sigma(\alpha)$ from p, p' , respectively, on opposite sides of η . It is easy to see that $|tw_\sigma(\delta_\alpha, \alpha)| \leq |tw_\sigma(\beta, \alpha)|$, where β is the geodesic through q, q' . Let r be the foot of the perpendicular as shown. As in the first part of the proof of Lemma 3.5 in [20], we note that since the images of $\tilde{\alpha}'$ under the translation along $\tilde{\alpha}$ are disjoint, the projection of $\tilde{\alpha}'$ on $\tilde{\alpha}$ has length at most $l_\sigma(\alpha)$. Thus $l(pr) < l(pq) = l(\alpha)$. Hence,

$$|tw_\sigma(\beta, \alpha)| = 2 \frac{l(pq) + l(pr)}{l(\alpha)} < 4. \quad \square$$

4.4. The product region theorem. Let $\mathcal{A} \subset \mathcal{S}$ be a collection of disjoint, homotopically distinct, simple closed curves on S and let $\mathcal{T}_{thin}(\mathcal{A}, \epsilon_0) \subset \mathcal{T}(S)$ be the subset in which all curves $\alpha \in \mathcal{A}$ have hyperbolic length at most ϵ_0 . Extend \mathcal{A} to a pants decomposition and define Fenchel-Nielsen coordinates as above. Let $S_{\mathcal{A}}$ denote the surface obtained from S by removing all the curves in \mathcal{A} and replacing the resulting boundary components by punctures.

Following [20], we now define a projection

$$\Pi: \mathcal{T}(S) \rightarrow \mathcal{T}(S_{\mathcal{A}}) \times \mathbb{H}_{\alpha_1} \times \dots \times \mathbb{H}_{\alpha_r},$$

where $\mathcal{A} = \{\alpha_1 \dots, \alpha_r\}$ and \mathbb{H}_{α_i} is the upper half-plane. The first component Π_0 which maps to $\mathcal{T}(S_{\mathcal{A}})$ is defined by forgetting the coordinates of the curves in \mathcal{A} and keeping the same Fenchel-Nielsen coordinates for the remaining surface. For $\alpha \in \mathcal{A}$ define $\Pi_\alpha: \mathcal{T}(S) \rightarrow \mathbb{H}_\alpha$ by

$$\Pi_\alpha(\sigma) = s_\alpha(\sigma) + i/l_\sigma(\alpha) \in \mathbb{H}_\alpha.$$

Let $d_{\mathbb{H}_\alpha}$ be half the usual hyperbolic metric on \mathbb{H}_α . Minsky's product region theorem states that, up to bounded additive error, Teichmüller distance on $\mathcal{T}_{thin}(\mathcal{A}, \epsilon_0)$ is equal to the sup metric on

$$\mathcal{T}(S_{\mathcal{A}}) \times \mathbb{H}_{\alpha_1} \times \dots \times \mathbb{H}_{\alpha_r}.$$

Theorem 4.5 (Minsky [20]). *Let $\sigma, \tau \in \mathcal{T}_{thin}(\mathcal{A}, \epsilon_0)$. Then*

$$d_{\mathcal{T}(S)}(\sigma, \tau) = \max_{\alpha \in \mathcal{A}} \{d_{\mathcal{T}(S_{\mathcal{A}})}(\Pi_0(\sigma), \Pi_0(\tau)), d_{\mathbb{H}_\alpha}(\Pi_\alpha(\sigma), \Pi_\alpha(\tau))\} \pm O(1).$$

We remark that Minsky makes several assumptions on the size of ϵ_0 in order to prove the above theorem. We may assume that our initial choice of ϵ_0 satisfies these assumptions. Recall from Section 2.1 that a curve $\alpha \in \mathcal{S}$ is said to be extremely short if $l_\sigma(\alpha) < \epsilon_0$. The distance between the projections to \mathbb{H}_α can be approximated as follows:

Lemma 4.6. *Suppose $\alpha \in \mathcal{S}$ is extremely short in both σ, τ . Then*

$$\exp 2d_{\mathbb{H}_\alpha}(\Pi_\alpha(\sigma), \Pi_\alpha(\tau)) \stackrel{*}{\asymp} \max \{l_\sigma l_\tau |s_\alpha(\sigma) - s_\alpha(\tau)|^2, l_\tau/l_\sigma, l_\sigma/l_\tau\}$$

where $l_\sigma = l_\sigma(\alpha)$ and $l_\tau = l_\tau(\alpha)$.

Proof. This is a simple calculation using the formula

$$\cosh 2d_{\mathbb{H}}(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2 \operatorname{Im} z_1 \operatorname{Im} z_2},$$

where $d_{\mathbb{H}}$ is as above half the usual hyperbolic distance in \mathbb{H} . □

The lemma reveals a useful fact from hyperbolic geometry: unless the difference $|x - x'|$ is extremely large, the distance between two points $x + iy, x' + iy' \in \mathbb{H}$ is dominated by $|\log[y/y']|$. In our situation this means that unless the difference between the twist coordinates is extremely large in comparison to the lengths, their contribution to hyperbolic distance can be neglected. We quantify this in the following useful corollary which shows that as long as each twist coordinate is bounded by $O(1/l)$, it can be neglected when estimating the contribution to Teichmüller distance coming from the same short curve in two surfaces.

Corollary 4.7. *Suppose that $\sigma_1, \sigma_2 \in \mathcal{T}_{thin}(\alpha, \epsilon_0)$ and that, for $i = 1, 2$, $\nu \in \mathcal{ML}(S)$ satisfies*

$$Tw_{\sigma_i}(\nu, \alpha) l_{\sigma_i}(\alpha) = O(1).$$

Then

$$d_{\mathbb{H}_\alpha}(\Pi_\alpha(\sigma_1), \Pi_\alpha(\sigma_2)) = \frac{1}{2} \left| \log \frac{l_{\sigma_1}(\alpha)}{l_{\sigma_2}(\alpha)} \right| \pm O(1).$$

Proof. This follows easily from Lemma 4.6, using Lemma 4.1 to approximate $s_\alpha(\sigma_1) - s_\alpha(\sigma_2)$ by $tw_{\sigma_1}(\nu, \alpha) - tw_{\sigma_2}(\nu, \alpha)$. (Cross terms such as $s_\alpha(\sigma_1)l_{\sigma_2}(\alpha)$ may be rearranged as $[s_\alpha(\sigma_1)l_{\sigma_1}(\alpha)][l_{\sigma_2}(\alpha)/l_{\sigma_1}(\alpha)]$.) Note that the multiplicative error in Lemma 4.6 translates to an additive error in the distance. \square

5. SHORT CURVES ALONG TEICHMÜLLER GEODESICS

In this section we prove Theorem B, stated more precisely as Theorem 5.10. This gives a combinatorial estimate for the hyperbolic length of an extremely short curve along the Teichmüller geodesic $\mathcal{G}(\nu^+, \nu^-)$. We also recall the estimate for the twist of ν^\pm around such curves (Theorem 5.11) proved in [22].

Deriving the length estimate is largely a matter of putting together results proved in [18, 21, 23]. Both estimates are made by a careful study of the relationship between the hyperbolic metric \mathcal{G}_t and the quadratic differential metric q_t . As indicated in the Introduction, there are two distinct reasons why a curve α may become extremely short: one is that the relative twisting of ν^+ and ν^- around α may be very large, the other that ν^+ and ν^- may have large relative complexity in $S \setminus \alpha$. We express the latter in terms of a scale factor which controls the relationship between the quadratic differential and hyperbolic metrics on the components of the thick part of \mathcal{G}_t adjacent to α , as made precise in Rafi's thick-thin decomposition for quadratic differentials, Theorem 5.5.

We begin in Sections 5.1 and 5.2 with some essential facts about the geometry of annuli with respect to quadratic differential metrics.

5.1. Annuli in quadratic differential metrics. We review the notions of *flat* and *expanding* annuli from [20]. These concepts provide the framework with which to analyze short curves.

Let $\Sigma \in \mathcal{T}(S)$ and let q be a quadratic differential on Σ . Let γ be a piecewise smooth curve in (S, q) . At a smooth point p , the curvature $\kappa(p)$ is well-defined, up to a choice of sign. If γ is the boundary component of a subsurface Y , we choose the sign to be positive if the acceleration vector at p points into Y . At a singular point P , although the curvature is not defined, we shall say γ is non-negatively curved at P if the interior angle $\theta(P)$ is at most π and say it is non-positively curved at P if $\theta(P)$ is at least π . By interior angle, we mean the angle that is on the same side of γ as Y . We say γ is monotonically curved

with respect to Y either if the curvature is non-negative at every point, or non-positive at every point. The total curvature of γ is given by

$$\kappa_Y(\gamma) = \int_{\gamma} \kappa(p) + \sum [\pi - \theta(P)],$$

where the sum is taken over all singular points P on γ . The Gauss-Bonnet theorem gives

$$(6) \quad \sum \kappa_Y(\gamma) - \pi \sum \text{ord} P = 2\pi\chi(Y),$$

where $\text{ord} P$ is the order of the zero at P , the first sum is over all boundary components γ of Y , and the second sum is over the zeros P of q in the interior of Y .

Let A be an annulus in (S, q) with piecewise smooth boundary. The following definitions are due to Minsky [18]. We say A is *regular* if both boundary components ∂_0, ∂_1 are monotonically curved with respect to A and if ∂_0, ∂_1 are q -equidistant from each other. Suppose that A is a regular annulus such that $\kappa_A(\partial_0) \leq 0$. We say A is an *expanding annulus* if $\kappa_A(\partial_0) < 0$ and we call ∂_0 the *inner* boundary and ∂_1 the *outer* boundary. Expanding annuli are exemplified by an annulus bounded by a pair of concentric circles in \mathbb{R}^2 . The inner boundary is the circle of smaller radius and has total curvature -2π , while the outer boundary has total curvature 2π .

A regular annulus is *primitive* with respect to q if it contains no singularities of q in its interior. By Equation (6), its boundaries satisfy $\kappa_A(\partial_0) = -\kappa_A(\partial_1)$. A regular annulus is *flat* if $\kappa_A(\partial_0) = \kappa_A(\partial_1) = 0$. By (6), a flat annulus is necessarily primitive, and is foliated by Euclidean geodesics homotopic to the boundaries. Thus a flat annulus is isometric to a cylinder obtained as the quotient of a Euclidean rectangle in \mathbb{R}^2 . Note that a primitive annulus must either be flat or expanding.

One reason for introducing flat and expanding annuli is that their moduli are easy to estimate. The following result can be deduced from Theorem 4.5 of [18] and is proved in [21]:

Theorem 5.1. *Let $A \subset S$ be an annulus that is primitive with respect to q and with inner and outer boundaries ∂_0 and ∂_1 , respectively. Let d be the q -distance between ∂_0 and ∂_1 . Then either*

- (i) *A is flat and $\text{Mod } A = d/l_q(\partial_0)$ or*
- (ii) *A is expanding and $\text{Mod } A \asymp \log[d/l_q(\partial_0)]$.*

5.2. Modulus of annulus and length of short curve. The link between the hyperbolic and quadratic differential metrics on a surface is made using annuli of large modulus. Let σ be the hyperbolic metric that uniformizes Σ . If α is short in σ , Maskit [14] showed that the

extremal length $Ext_\Sigma(\alpha)$ and hyperbolic length $l_\sigma(\alpha)$ are comparable, up to multiplicative constants. Moreover, there is an embedded collar $C(\alpha)$ around α in (S, σ) whose modulus is comparable to $1/l_\sigma(\alpha)$ (see [20] for an explicit calculation), and therefore also to $1/Ext_\Sigma(\alpha)$. By the (geometric) definition of extremal length, this implies that the maximal annulus around α in Σ has modulus comparable to $1/l_\sigma(\alpha)$. The following theorem of Minsky allows us to replace any annulus of sufficiently large modulus with one that is primitive:

Theorem 5.2. ([18] Theorem 4.6) *Let $A \subset \Sigma$ be any homotopically non-trivial annulus whose modulus is sufficiently large and let $q \in \mathcal{Q}(\Sigma)$. Then A contains an annulus B that is primitive with respect to q and such that $\text{Mod } A \asymp \text{Mod } B$.*

(The statement of Theorem 4.6 in [18] should read $\text{Mod } A \geq m_0$ not $\text{Mod } A \leq m_0$.) Thus, we have

Theorem 5.3. *If α is a simple closed curve which is sufficiently short in (S, σ) , then for any $q \in \mathcal{Q}(\Sigma)$, there is an annulus A that is primitive with respect to q with core homotopic to α such that*

$$\frac{1}{l_\sigma(\alpha)} \asymp \text{Mod}(A).$$

We may assume that ϵ_0 was chosen so that if $l_\sigma(\alpha) < \epsilon_0$, then $l_\sigma(\alpha)$ is small enough that this theorem is valid.

We can now apply Theorem 5.1 in the following way. It follows from Equation (6) that every simple closed curve γ on (S, q) either has a unique q -geodesic representative, or is contained in a family of closed Euclidean geodesics which foliate a flat annulus [26]. Denote by $F(\gamma)$ the maximal flat annulus, which necessarily contains all q -geodesic representatives of γ . If the geodesic representative of γ is unique, then $F(\gamma)$ is taken to be the degenerate annulus containing this geodesic alone. Denote the (possibly coincident) boundary curves of $F(\gamma)$ by ∂_0, ∂_1 and consider the q -equidistant curves from ∂_i outside $F(\gamma)$. Let $\hat{\partial}_i$ denote the first such curve which is not embedded. If $\hat{\partial}_i \neq \partial_i$, then the pair $\partial_i, \hat{\partial}_i$ bounds a region $E_i(\gamma)$ whose interior is an annulus with core homotopic to γ , and which is by its construction regular and expanding. Combining the preceding two theorems with Theorem 5.1 we have:

Corollary 5.4. *If α is an extremely short curve on (S, σ) , then*

$$\frac{1}{l_\sigma(\alpha)} \asymp \max \{ \text{Mod } F(\alpha), \text{Mod } E_0(\alpha), \text{Mod } E_1(\alpha) \}.$$

Proof. Since α is extremely short, we have $1/Ext_\Sigma(\alpha) \asymp 1/l_\sigma(\alpha)$ and hence it follows from the (geometric) definition of extremal length that

$$\frac{1}{l_\sigma(\alpha)} \asymp \frac{1}{Ext_\Sigma(\alpha)} \geq \max \{ \text{Mod } F(\alpha), \text{Mod } E_0(\alpha), \text{Mod } E_1(\alpha) \}.$$

By Theorem 5.3, there is a primitive annulus A whose core is homotopic to α such that $1/l_\sigma(\alpha) \asymp \text{Mod}(A)$. We will show that

$$\text{Mod}(A) \prec \max \{ \text{Mod } F(\alpha), \text{Mod } E_0(\alpha), \text{Mod } E_1(\alpha) \}.$$

If A is flat, then A must be contained in the maximal flat annulus $F(\alpha)$. In this case, $\text{Mod}(A) \leq \text{Mod } F(\alpha)$. If A is expanding, then although it may not be contained in either $E_0(\alpha)$ or $E_1(\alpha)$ it must be disjoint from the interior of $F(\alpha)$. Without loss of generality, let us assume that A lies on the same side of $F(\alpha)$ as $E_0(\alpha)$. Let ∂_0 and $\hat{\partial}_0$ be respectively, the inner and outer boundaries of $E_0(\alpha)$. Let C_0 and C_1 be respectively, the inner and outer boundaries of A . Since $l_q(\partial_0)$ is equal to the q -length of the geodesic representative of α , the q -length of the inner boundary of A satisfies $l_q(C_0) \geq l_q(\partial_0)$. Let ω be a q -shortest arc in $E_0(\alpha)$ from ∂_0 to itself; its length is $2d_q(\partial_0, \hat{\partial}_0)$. The intersection $\omega \cap A$ is a union of two arcs, each of which goes from one boundary component of A to another. Since $l_q(\omega \cap A) \leq l_q(\omega)$, it follows that $d_q(C_0, C_1) \leq l_q(\omega \cap A)/2 \leq d_q(\partial_0, \hat{\partial}_0)$. Thus, it follows from Theorem 5.1 that

$$\text{Mod}(A) \asymp \log \frac{d_q(C_0, C_1)}{l_q(C_0)} \leq \log \frac{d_q(\partial_0, \hat{\partial}_0)}{l_q(\partial_0)} \asymp \text{Mod}(E_0(\alpha)). \quad \square$$

The idea of our basic length estimates for extremely short curves α in Theorem 5.10, is to combine this corollary with the estimates for the moduli of $F(\alpha)$ and $E_i(\alpha)$ in Theorem 5.1.

5.3. Thick-thin decomposition and the q -metric. The thick-thin decomposition for quadratic differentials developed in [23] describes the relationship between the q -metric on the surface Σ and the uniformizing hyperbolic metric σ in the thick components of the thick-thin decomposition of σ . It states that on the hyperbolic thick parts of (S, σ) the two metrics are comparable, up to a factor which depends on the moduli of the expanding annuli around the short curves in the boundary of the thick component. This factor will be crucial in our estimates below.

To make a precise statement, for a subsurface Y of S , define the q -geodesic representative of Y to be the unique subsurface \hat{Y} of (S, q) with q -geodesic boundary in the homotopy class of Y that is disjoint

from the interior of $F(\gamma_i)$ for all components γ_i of ∂Y . Notice that \hat{Y} is q -geodesically convex, so that if a closed curve ζ is contained in Y , it has a q -geodesic representative contained in \hat{Y} . (It is possible for \hat{Y} to be degenerate. See [23] for an example where the area of \hat{Y} is zero.)

If Y is not a pair of pants, define λ_Y to be the length of the q -shortest non-peripheral simple closed curve contained in \hat{Y} . If Y is a pair of pants, define λ_Y to be $\max\{l_q(\gamma_1), l_q(\gamma_2), l_q(\gamma_3)\}$ where $\gamma_1, \gamma_2, \gamma_3$ are the three boundary curves of \hat{Y} . The thick-thin decomposition for quadratic differentials is the following:

Theorem 5.5 (Rafi [23]). *Let σ be the hyperbolic metric that uniformizes Σ and let Y be a thick component of the hyperbolic thick-thin decomposition of (S, σ) . Then*

- (i) $\text{diam}_q \hat{Y} \asymp \lambda_Y$,
- (ii) *For any non-peripheral simple closed curve ζ in Y , we have*

$$l_q(\zeta) \asymp \lambda_Y l_\sigma(\zeta).$$

5.4. Twist in the q -metric. In order to compare two surfaces, we need to estimate not only the lengths but also the twist parameters of short curves. To do this we use a signed version of Rafi's definition ([22] Section 4) of the twist of a simple curve ζ about another curve α in a quadratic differential metric q on S .

Let $A \subset S$ be a regular annulus with core curve α , let \tilde{S} be the annular cover of S corresponding to α , and let $\tilde{\alpha}$ be a \tilde{q} -geodesic representative of the core of \tilde{S} . Suppose that ζ is a simple q -geodesic (i.e. geodesic with respect to the q -metric) that intersects α , and let $\tilde{\zeta}$ be a lift which intersects $\tilde{\alpha}$. Let $\tilde{\beta}$ be a bi-infinite \tilde{q} -geodesic arc in \tilde{S} that is orthogonal to $\tilde{\alpha}$. We would like to define the twist $tw_q(\zeta, \alpha)$ to be the sum $a_{\tilde{S}}(\tilde{\zeta}, \tilde{\beta})$ of the signed intersection numbers over all intersections between $\tilde{\zeta}$ and $\tilde{\beta}$. The following lemma shows that $a_{\tilde{S}}(\tilde{\zeta}, \tilde{\beta})$ is, up to a bounded additive error, independent of the choices of $\tilde{\beta}$ and $\tilde{\zeta}$.

Lemma 5.6. *Let α , $\tilde{\alpha}$, and \tilde{S} be as above and suppose that ζ is a simple q -geodesic transverse to α . Suppose that $\tilde{\zeta}, \tilde{\zeta}'$ are different lifts of ζ that intersect $\tilde{\alpha}$ and that $\tilde{\beta}, \tilde{\beta}'$ are different bi-infinite \tilde{q} -geodesic arcs orthogonal to $\tilde{\alpha}$. Then*

$$a_{\tilde{S}}(\tilde{\zeta}, \tilde{\beta}) = a_{\tilde{S}}(\tilde{\zeta}', \tilde{\beta}') \pm O(1).$$

Proof. Let $\tilde{F}(\alpha)$ be the lift to \tilde{S} of the maximal flat annulus $F(\alpha)$ around $\tilde{\alpha}$ and let $a_F(\tilde{\zeta}, \tilde{\beta})$ denote the sum of signed intersection numbers over intersection points within $\tilde{F}(\alpha)$. A simple Euclidean argument shows that for any two disjoint arcs $\tilde{\zeta}, \tilde{\zeta}'$, we have

$$a_F(\tilde{\zeta}, \tilde{\beta}) = a_F(\tilde{\zeta}', \tilde{\beta}') \pm O(1).$$

We claim that outside $\tilde{F}(\alpha)$, any two \tilde{q} -geodesics can intersect at most twice. Outside $\tilde{F}(\alpha)$, \tilde{S} is made up of two regular expanding annuli E_1, E_2 , one attached to each boundary of $\tilde{F}(\alpha)$. These annuli extend out to infinity in \tilde{S} (which can be compactified using the hyperbolic metric on S , see [17] Section 2.4). The key point is that in any expanding annulus E , two geodesic arcs can intersect at most once. For if they intersected twice, we would get a piecewise geodesic loop γ homotopic to the inner boundary, made up of two geodesic arcs that go from one intersection point to the other. Along each arc, the geodesic curvature vanishes. The Gauss-Bonnet theorem in Equation (6) applied to the annulus bounded by γ and the inner boundary shows this is impossible. (Notice that if E is not primitive, then the singularities of q in E only improve the desired inequality in Equation (6).) \square

We define $tw_q(\zeta, \alpha)$ to be the minimum of the numbers $a_{\tilde{S}}(\tilde{\zeta}, \tilde{\beta})$ over all choices of $\tilde{\zeta}, \tilde{\beta}$. Notice that the argument requires only that the lifts of ζ to \tilde{S} be disjoint, so that we can similarly define $tw_q(\nu, \alpha)$ for a geodesic lamination ν where as usual, $tw_q(\nu, \alpha)$ depends only the underlying support $|\nu|$ of ν .

The following key result allows us directly to compare the twists in the hyperbolic and quadratic differential metrics.

Proposition 5.7 ([22] Theorem 4.3). *Suppose that σ is a hyperbolic metric uniformizing a surface $\Sigma \in \mathcal{T}(S)$ and that $q \in \mathcal{Q}(\Sigma)$, and let ν be a geodesic lamination intersecting α . Then*

$$|tw_\sigma(\nu, \alpha) - tw_q(\nu, \alpha)| = O(1/l_\sigma(\alpha)).$$

The statement in [22] has $i_\alpha(\nu, \delta_\alpha)$ in place of $tw_\sigma(\nu, \alpha)$, however by Lemma 4.4, this distinction is unimportant. The result in [22] is stated for closed curves but extends immediately to the case of geodesic laminations as explained above.

5.5. Length and twist along $\mathcal{G}(\nu^+, \nu^-)$. As explained at the end of Section 5.2, we can use Theorem 5.1 and Corollary 5.4 to estimate the length of an extremely short curve α in \mathcal{G}_t . We call a flat or expanding annulus which achieves the maximum modulus in Corollary 5.4 a *dominant annulus* for α . There may be more than one dominant

annulus, but this will not affect our reasoning and we will refer to ‘the dominant annulus’. The estimates depend on whether the dominant annulus is flat or expanding, corresponding to the two terms $D_t(\alpha)$ and $K_t(\alpha)$ in the main result Theorem 5.10 of this section.

Suppose first that the maximal flat annulus $F_t(\alpha)$ is dominant. Provided that α is neither vertical nor horizontal (see Section 2.8), the following proposition expresses $\text{Mod } F_t(\alpha)$ in terms of the relative twisting $d_\alpha(\nu^+, \nu^-)$ of ν^+, ν^- around α defined in Section 4.3. The case in which α is either horizontal or vertical, so that either ν^+ or ν^- has empty intersection with α , is easier and is dealt with in Section 5.6.

Proposition 5.8. *Let α be a curve in (S, q) that is neither vertical nor horizontal and suppose that the maximal flat annulus $F_t(\alpha)$ is dominant. Then*

$$\text{Mod } F_t(\alpha) \asymp e^{-2|t-t_\alpha|} d_\alpha(\nu^+, \nu^-).$$

Proof. Since a flat annulus is Euclidean, its geometry is very simple. Let η be a q_t -geodesic arc in $F_t(\alpha)$ joining the two boundaries of $F_t(\alpha)$ that is orthogonal to the geodesic representatives of α . For a simple geodesic ζ transverse to α , define $tw_{F_t}(\zeta, \alpha)$ to be the signed intersection number of ζ with η in $F_t(\alpha)$. It is independent of the choice of η up to a bounded error of 1. Assuming that α is neither vertical nor horizontal, then at the balance time t_α (see Section 2.8) the horizontal and vertical foliations both make an angle of $\pi/4$ with the q_{t_α} -geodesic representatives of α . In this case, a leaf of $\nu_{t_\alpha}^+$ or $\nu_{t_\alpha}^-$ intersects η approximately (up to an error of 1) $l_{q_{t_\alpha}}(\eta)/l_{q_{t_\alpha}}(\alpha)$ times, so the modulus of $F_{t_\alpha}(\alpha)$ is approximated by $tw_{F_{t_\alpha}}(\nu^+, \alpha) = tw_{F_{t_\alpha}}(\nu^-, \alpha)$. More generally, the horizontal leaves make an angle ψ_t with α , where $|\tan \psi_t| = e^{2(t-t_\alpha)}$. From this it is a straightforward exercise in Euclidean geometry, see Section 4.1 of [22], to prove:

$$(7) \quad |tw_{F_t}(\nu^\pm, \alpha) - e^{\mp 2(t-t_\alpha)} \text{Mod } F_t(\alpha)| \leq 1.$$

We will show that

$$(8) \quad |tw_{F_t}(\nu^+, \alpha) - tw_{F_t}(\nu^-, \alpha)| = d_\alpha(\nu^+, \nu^-) \pm O(1),$$

from which the proposition follows. From the proof of Lemma 5.6 we have $tw_{q_t}(\zeta, \alpha) = tw_{F_t}(\zeta, \alpha) \pm O(1)$. Now it was observed in Section 4.2 (see also the subsequent discussion in Section 4.3) that although $tw_\sigma(\nu, \alpha)$ depends on the metric σ in which it is measured, the difference in twist of ν^+ and ν^- equals (up to a bounded error) the number of times a leaf of ν^+ intersects a leaf of ν^- in the annular cover of S corresponding to α . Since this also holds for a quadratic

differential metric, we get:

$$\begin{aligned} |tw_{q_t}(\nu^+, \alpha) - tw_{q_t}(\nu^-, \alpha)| &= |tw_\sigma(\nu^+, \alpha) - tw_\sigma(\nu^-, \alpha)| \pm O(1) \\ &= d_\alpha(\nu^+, \nu^-) \pm O(1). \end{aligned}$$

Equation (8) follows. \square

Suppose now that one or other of the expanding annuli around α is dominant. The estimate of modulus in this case is given by:

Proposition 5.9. *Let $q \in \mathcal{Q}(\Sigma)$. Suppose that α is extremely short in σ and let Y be a thick component of the hyperbolic thick-thin decomposition of (S, σ) , one of whose boundary components is α . Let $\hat{\alpha}$ be the q -geodesic representative of α on the boundary of \hat{Y} and let $E(\alpha)$ be a maximal expanding annulus on the same side of $\hat{\alpha}$ as \hat{Y} . If $E(\alpha)$ is dominant, then*

$$\text{Mod } E(\alpha) \asymp \log \frac{\lambda_Y}{l_q(\alpha)}.$$

Proof. Let d_q denote the q -metric. By Theorems 5.2 and 5.1(ii), it is sufficient to show that $d_q(\partial_0, \partial_1) \asymp \lambda_Y$.

Note that although $E(\alpha)$ is not necessarily contained in \hat{Y} , the outer boundary ∂_1 must intersect \hat{Y} . Hence, $d_q(\partial_0, \partial_1) \leq \text{diam}_q(\hat{Y})$ and so by Theorem 5.5(i), we have $d_q(\partial_0, \partial_1) \prec \lambda_Y$.

Now we prove the inequality in the other direction. Observe that since $E(\alpha)$ is maximal, the outer boundary ∂_1 intersects itself and so there is a non-trivial arc ω with endpoints on $\hat{\alpha}$ whose length is $2d_q(\partial_0, \partial_1)$. First suppose that ω is contained in \hat{Y} . A regular neighborhood of $\hat{\alpha} \cup \omega$ is a pair of pants whose boundary curves are homotopic to α and two additional curves ζ_1, ζ_2 . Note that for $i = 1, 2$,

$$l_q(\omega) + l_q(\alpha) \geq l_q(\zeta_i).$$

Thus, if either ζ_1 or ζ_2 , say ζ_1 is non-peripheral in Y , then

$$\frac{d_q(\partial_0, \partial_1)}{l_q(\alpha)} \succ \frac{l_q(\zeta_1)}{l_q(\alpha)} \geq \frac{\lambda_Y}{l_q(\alpha)}.$$

If both ζ_1, ζ_2 are peripheral, then Y is a pair of pants, and we have

$$\frac{d_q(\partial_0, \partial_1)}{l_q(\alpha)} \succ \frac{l_q(\zeta_1) + l_q(\zeta_2)}{l_q(\alpha)} \asymp \frac{\lambda_Y}{l_q(\alpha)}.$$

If ω exits \hat{Y} , we replace it with a new arc ω' as follows. Let p be the first exit point and let γ be the boundary component of \hat{Y} that contains p . Let ω' be the arc that first goes along ω to p , then makes one turn around γ from p to itself, then comes back to $\hat{\alpha}$ along the

first path. Because γ is in the boundary of \hat{Y} , its hyperbolic length is extremely short and thus by Theorem 5.3, the original arc ω must pass through an annulus of large modulus with core curve γ . Therefore, $l_q(\gamma) \prec l_q(\omega)$ and so we have

$$l_q(\omega') \leq 2l_q(\omega) + l_q(\gamma) \prec l_q(\omega).$$

Now we can run the same argument as above with ω' in place of ω to deduce the desired inequality. \square

We are now able to write down the desired length estimate. Suppose that the curve α is extremely short in some surface \mathcal{G}_t along the Teichmüller geodesic $\mathcal{G}(\nu^+, \nu^-)$. Let Y_1, Y_2 be the thick components of the thick-thin decomposition of (S, \mathcal{G}_t) that are adjacent to α (where Y_1 may equal Y_2). Define

$$(9) \quad K_t(\alpha) = \max \left\{ \frac{\lambda_{Y_1}}{l_{\mathcal{G}_t}(\alpha)}, \frac{\lambda_{Y_2}}{l_{\mathcal{G}_t}(\alpha)} \right\}$$

and

$$(10) \quad D_t(\alpha) = e^{-2|t-t_\alpha|} d_\alpha(\nu^+, \nu^-).$$

Then, combining Corollary 5.4 and Propositions 5.8 and 5.9, we obtain our first main result which is essentially Theorem B of the Introduction:

Theorem 5.10. *Let α be a curve on S that is neither vertical nor horizontal. If α is extremely short in \mathcal{G}_t , then*

$$\frac{1}{l_{\mathcal{G}_t}(\alpha)} \asymp \max\{D_t(\alpha), \log K_t(\alpha)\}.$$

It was shown in [21] that $K_t(\alpha)$ can be estimated combinatorially as the subsurface intersection of ν^+, ν^- in the corresponding component of the thick part of \mathcal{G}_t adjacent α . Since this is not necessary for our development, we shall not go into this here.

We need to estimate not only the lengths but also the twist parameters about short curves. Combining Proposition 5.7 with Equation (7), Lemma 5.6, Proposition 5.8, and Theorem 5.10, we get:

Theorem 5.11 ([22] Theorem 1.3). *With the same hypotheses as Theorem 5.10,*

$$\begin{aligned} Tw_{\mathcal{G}_t}(\nu^+, \alpha) &= O\left(\frac{1}{l_{\mathcal{G}_t}(\alpha)}\right) \text{ if } t \geq t_\alpha, \\ Tw_{\mathcal{G}_t}(\nu^-, \alpha) &= O\left(\frac{1}{l_{\mathcal{G}_t}(\alpha)}\right) \text{ if } t \leq t_\alpha. \end{aligned}$$

5.6. Estimates for horizontal and vertical short curves on \mathcal{G} .

We also need the analogue of Theorems 5.10 and 5.11 for extremely short curves α which are either horizontal or vertical.

For definiteness, assume α is vertical so that $i(\alpha, \nu^-) = 0$. The definition of balance time no longer makes sense. Instead, we work relative to time $t = 0$. Let d be the height (i.e., distance between the two boundaries) of $F_0(\alpha)$. At an arbitrary time t ,

$$l_{q_t}(\alpha) = i(\alpha, \nu_t^+) = e^t l_{q_0}(\alpha)$$

while the height of $F_t(\alpha)$ is $e^{-t}d$. Hence

$$(11) \quad \text{Mod } F_t(\alpha) = e^{-2t} \text{Mod } F_0(\alpha).$$

This is the analogue of Proposition 5.8.

If α is vertical, the discussion in Proposition 5.9 about expanding annuli is unchanged. Thus we obtain:

Theorem 5.12. *Let α be a vertical curve on S . If α is extremely short in \mathcal{G}_t , then*

$$\frac{1}{l_{\mathcal{G}_t}(\alpha)} \asymp \max \{ e^{-2t} \text{Mod } F_0(\alpha), \log K_t(\alpha) \}.$$

If α is horizontal, the estimate is the same except that the first term is replaced by $e^{2t} \text{Mod } F_0(\alpha)$.

We also want the analogue of Theorem 5.11. If α is vertical, then $Tw_{\mathcal{G}_t}(\nu^-, \alpha)$ is undefined. However we have:

Theorem 5.13. *If α is extremely short in \mathcal{G}_t , and if α is vertical then*

$$Tw_{\mathcal{G}_t}(\nu^+, \alpha) = O \left(\frac{1}{l_{\mathcal{G}_t}(\alpha)} \right),$$

while if α is horizontal then

$$Tw_{\mathcal{G}_t}(\nu^-, \alpha) = O \left(\frac{1}{l_{\mathcal{G}_t}(\alpha)} \right).$$

Proof. If α is vertical, the q -twist $tw_q(\nu^+, \alpha)$ in $F_t(\alpha)$ vanishes, while if α is horizontal, then $tw_q(\nu^-, \alpha) = 0$. The result follows from Lemma 5.6 and Proposition 5.7. \square

6. SHORT CURVES ALONG LINES OF MINIMA

In this section we prove Theorem B, stated more precisely as Theorem 6.1, which gives our combinatorial estimate for the length of a curve which becomes extremely short at some point along the line of

minima $\mathcal{L}(\nu^+, \nu^-)$. We also estimate the twist of ν^\pm around α in Theorem 6.2. This will form the basis for our comparison of the metrics \mathcal{L}_t and \mathcal{G}_t . It turns out that, in a close parallel to the case of the Teichmüller geodesic, there are two reasons why a curve can be extremely short: either the relative twisting of ν^+, ν^- about α is large, or one or other of the pants curves in a pair of pants adjacent to α in a short marking of \mathcal{L}_t has large intersection with either ν^+ or ν^- .

More precisely, suppose that α is an extremely short curve in \mathcal{L}_t and let $\mathcal{P}_{\mathcal{L}_t}$ be a short pants system in \mathcal{L}_t , which necessarily contains α . Define

$$(12) \quad H_t(\alpha) = \sup_{\beta \in \mathcal{B}} \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)},$$

where \mathcal{B} is the set of pants curves in $\mathcal{P}_{\mathcal{L}_t}$ which are boundaries of pants adjacent to α and q_t is the quadratic differential metric of area 1 (on the corresponding surface \mathcal{G}_t) whose horizontal and vertical foliations are respectively, ν_t^+ and ν_t^- .

Let $D_t(\alpha)$ be as in Equation (10). Our main estimates are:

Theorem 6.1. *Let α be a curve on S which is neither vertical nor horizontal. If α is extremely short in \mathcal{L}_t , then*

$$\frac{1}{l_{\mathcal{L}_t}(\alpha)} \asymp \max \left\{ D_t(\alpha), \sqrt{H_t(\alpha)} \right\}.$$

Theorem 6.2. *With the same hypotheses as Theorem 6.1, the twist satisfies:*

$$\begin{aligned} Tw_{\mathcal{L}_t}(\nu^+, \alpha) &= O\left(\frac{1}{l_{\mathcal{L}_t}(\alpha)}\right) \quad \text{if } t \geq t_\alpha, \\ Tw_{\mathcal{L}_t}(\nu^-, \alpha) &= O\left(\frac{1}{l_{\mathcal{L}_t}(\alpha)}\right) \quad \text{if } t \leq t_\alpha. \end{aligned}$$

To prove Theorems 6.1, 6.2, we note that since the surface \mathcal{L}_t is on the line of minima, we have at the point \mathcal{L}_t ,

$$(13) \quad dl(\nu_t^+) + dl(\nu_t^-) = 0.$$

The pants curves in $\mathcal{P}_{\mathcal{L}_t}$ (together with seams) define a set of coordinates $(l_\sigma(\alpha), \tau_\alpha(\sigma))$ on $\mathcal{T}(S)$ as explained in Section 4.2, which in turn define infinitesimal twist $\frac{\partial}{\partial \tau_\alpha}$ and length $\frac{\partial}{\partial l(\alpha)}$ deformations for $\alpha \in \mathcal{P}_{\mathcal{L}_t}$. Theorems 6.1 and 6.2 will follow from the relations we get by applying Equation (13) to $\frac{\partial}{\partial \tau_\alpha}$ and $\frac{\partial}{\partial l(\alpha)}$. For $\frac{\partial}{\partial \tau_\alpha}$, we use the well-known formula of Kerckhoff [11] and Wolpert [27], while for $\frac{\partial}{\partial l(\alpha)}$ we use the analogous formula for the length deformation derived in [24].

6.1. Differentiation with respect to twist. Suppose as above that α is an extremely short curve in \mathcal{L}_t . If we apply Equation (13) to $\frac{\partial}{\partial \tau_\alpha}$, the derivative formula in [11] and [27] gives

$$0 = \frac{\partial l(\nu_t^+)}{\partial \tau_\alpha} + \frac{\partial l(\nu_t^-)}{\partial \tau_\alpha} = \int_\alpha \cos \theta^+ d\nu_t^+ + \int_\alpha \cos \theta^- d\nu_t^-,$$

where θ^\pm is the function measuring the angle from each arc of $|\nu_t^\pm|$ to α . Assume that α is neither vertical nor horizontal, so that neither $i(\nu^+, \alpha)$ nor $i(\nu^-, \alpha)$ is zero. Then we may define the average angle Θ_t^\pm by

$$\cos \Theta_t^\pm = \frac{1}{i(\nu_t^\pm, \alpha)} \int_\alpha \cos \theta^\pm d\nu_t^\pm.$$

Setting $T = t - t_\alpha$, the preceding two equations give

$$(14) \quad e^T \cos \Theta_t^+ + e^{-T} \cos \Theta_t^- = 0.$$

If a particular leaf L of a lamination $|\nu|$ cuts α at an angle θ at a point p , then from the definition of the twist (see Section 4.1) and simple hyperbolic geometry we have

$$\cos \theta = \tanh \frac{tw_{\mathcal{L}_t}(L, \alpha, p) l_{\mathcal{L}_t}(\alpha)}{2}.$$

Since the twists $tw_{\mathcal{L}_t}(L, \alpha, p)$ for different leaves L differ by at most 1, if α is sufficiently short we obtain the estimate

$$(15) \quad |\cos \theta - \cos \Theta_t^\pm| = O(l_{\mathcal{L}_t}(\alpha))$$

from which we deduce that either $\cos \Theta_t^\pm$ and $tw_{\mathcal{L}_t}(\nu^\pm, \alpha)$ have the same sign, or that $|\cos \Theta_t^\pm| = O(l_{\mathcal{L}_t}(\alpha))$ so that $|tw_{\mathcal{L}_t}(\nu^\pm, \alpha)| = O(1)$.

Note also that Equation (14) implies that ν^+, ν^- twist around α in opposite directions and that the lamination whose weight on α is smaller does more of the twisting.

6.2. Differentiation with respect to length. For the length deformation, we shall apply the extension of the Wolpert formula derived in [24], which gives a general expression for $dl(\zeta)$, for $\zeta \in \mathcal{S}$, with reference to a pants curves system \mathcal{P} . Let $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n$ be the lifts of the pants curves in \mathcal{P} successively met by ζ , where the segment of the lift $\tilde{\zeta}$ of ζ between $\tilde{\alpha}_1$ and $\tilde{\alpha}_n$ projects to one complete period of ζ . Let d_j be the length of the common perpendicular π_j between $\tilde{\alpha}_j, \tilde{\alpha}_{j+1}$ and let S_j be the signed distance between π_{j-1} and π_j along $\tilde{\alpha}_j$, where the sign is positive if the direction from π_{j-1} to π_j coincides with the orientation of α_j . (Note that if $\tilde{\alpha}_j$ and $\tilde{\alpha}_{j+1}$ project to the same curve α and if α is adjacent to two distinct pairs of pants, then π_j projects to

an arc perpendicular to α that is not a seam.) Then Equation (3) of [24] states that

$$(16) \quad dl(\zeta) = \sum_{j=1}^n \cosh u_j dd_j + \sum_{j=1}^n \cos \theta_j dS_j,$$

where θ_j is the angle from $\tilde{\zeta}$ to $\tilde{\alpha}_j$ measured counter-clockwise and u_j is the complex distance from $\tilde{\zeta}$ to the complete bi-infinite geodesic which contains π_j . Replacing sums by integrals, we see that this formula, derived in [24] for closed curves, pertains equally to a measured lamination.

In our case, we take \mathcal{P} to be $\mathcal{P}_{\mathcal{L}_t}$ and apply this formula to $\frac{\partial}{\partial l(\alpha)}$ for $\alpha \in \mathcal{P}_{\mathcal{L}_t}$. The non-zero contributions will be from terms dS_j corresponding to lifts of α , and from two types of terms dd_j : those corresponding to perpendiculars with endpoints on lifts of α , and those corresponding to perpendiculars which do not intersect any lift of α , but whose projections are contained in a common pair of pants with α .

We first estimate the contribution from the terms dd_j . Suppose as above that $\mathcal{P}_{\mathcal{L}_t}$ is a short pants decomposition for \mathcal{L}_t . Let P be a pair of pants in $S \setminus \mathcal{P}_{\mathcal{L}_t}$ that has α as a boundary component. The geometry of P is completely determined by the lengths of the three boundary curves α, β, γ . A common perpendicular joining two (not necessarily distinct) boundary components of P may or may not have one of its endpoints on a boundary curve which projects to α on S . We say that the common perpendiculars of the first kind are *adjacent* to α , while those of the second type are not. The terms dd_j are estimated by the following lemma which is proved in the Appendix:

Lemma 6.3. *Suppose that α is extremely short in \mathcal{L}_t and let P be a pair of pants in $S \setminus \mathcal{P}_{\mathcal{L}_t}$ that has α as a boundary component. Let v denote the length of a common perpendicular adjacent to α , and let w denote the length of a common perpendicular not adjacent to α . Then*

$$\frac{\partial v}{\partial l(\alpha)} \stackrel{*}{\asymp} -\frac{1}{l(\alpha)} \quad \text{and} \quad \frac{\partial w}{\partial l(\alpha)} \stackrel{*}{\asymp} l(\alpha),$$

where the partial derivatives are taken with respect to the coordinates $(l(\alpha), \tau_\alpha)_{\alpha \in \mathcal{P}_{\mathcal{L}_t}}$.

We remark that the first of these estimates coincides with the heuristic computation that since the collar around α has length comparable to $\log[1/l(\alpha)]$, the derivative should be approximately $-1/l(\alpha)$.

We also need to bound the coefficient $\cosh u_j$ of dd_j in Equation (16). Notice that the bound applies to all the pants curves (not just the extremely short ones) in a short marking.

Lemma 6.4. *If the pants curves system \mathcal{P} is short, then for all j ,*

$$|\cosh u_j| \stackrel{*}{\gtrsim} 1.$$

Proof. Since \mathcal{P} is short, by definition all curves α_j have length bounded above, and hence the length d_j of the common perpendicular π_j to $\tilde{\alpha}_j, \tilde{\alpha}_{j+1}$ is bounded below.

First suppose that $\tilde{\zeta}$ intersects the infinite geodesic $\hat{\pi}_j$ that contains π_j . In this case, $u_j = i\phi_j$, where ϕ_j is the angle between $\tilde{\zeta}$ and $\hat{\pi}_j$ at their intersection point o . Consider the case when o is contained in the segment π_j . Let x_j be the distance between o and the endpoint o_j of π_j that lies on $\tilde{\alpha}_j$. Since $\tilde{\zeta}$ intersects both $\tilde{\alpha}_j$ and $\tilde{\alpha}_{j+1}$, the angle of parallelism formula gives

$$|\tan \phi_j| < 1/\sinh x_j \quad \text{and} \quad |\tan \phi_j| < 1/\sinh(d_j - x_j).$$

Since at least one of x_j and $d_j - x_j$ is bounded below, this gives a uniform upper bound on $|\tan \phi_j|$. Thus ϕ_j is uniformly bounded away from $\pi/2$ and $|\cosh u_j| = |\cos \phi_j|$ is bounded below by a universal positive number.

Now consider the case when o lies outside of π_j . Let o'_j, o'_{j+1} denote respectively, the points of intersection between $\tilde{\zeta}_j$ and $\tilde{\alpha}_j, \tilde{\alpha}_{j+1}$ and let o_j, o_{j+1} denote respectively, the points of intersection between π_j and $\tilde{\alpha}_j, \tilde{\alpha}_{j+1}$. If $d(o, o_j) \geq d(o, o_{j+1})$, then replace $\tilde{\zeta}$ with the geodesic $\tilde{\zeta}'$ that passes through o'_j and o_{j+1} and if $d(o, o_j) \leq d(o, o_{j+1})$, then replace $\tilde{\zeta}$ with the geodesic $\tilde{\zeta}'$ that passes through o'_{j+1} and o_j . The angle ϕ'_j of intersection between $\tilde{\zeta}'$ and π_j satisfies $|\cos \phi_j| > |\cos \phi'_j|$. We now run the preceding argument with $\tilde{\zeta}'$ in place of $\tilde{\zeta}$ to conclude $|\cos \phi'_j|$ is bounded below.

Now, suppose that $\tilde{\zeta}$ does not intersect $\hat{\pi}_j$. Then u_j is the hyperbolic distance from $\tilde{\zeta}$ to $\hat{\pi}_j$. Denote by p the point where the common perpendicular from $\tilde{\zeta}$ to $\hat{\pi}_j$ meets $\hat{\pi}_j$; this point may lie outside the segment π_j between $\tilde{\alpha}_j, \tilde{\alpha}_{j+1}$. Let y_j, y'_j denote the (unsigned) distances from p to $\tilde{\alpha}_j, \tilde{\alpha}_{j+1}$ respectively. The quadrilateral formula gives $\sinh y_j \sinh u_j = |\cos \theta_j|$ and $\sinh y'_j \sinh u_j = |\cos \theta_{j+1}|$, where θ_j, θ_{j+1} are the angles between $\tilde{\zeta}$ and $\tilde{\alpha}_j, \tilde{\alpha}_{j+1}$ respectively. Whether or not $p \in \pi_j$, at least one of y_j and y'_j is bounded below by $d_j/2$. Since there is a uniform lower bound on d_j , it follows that $|\sinh u_j|$ and hence $|\cosh u_j|$ is uniformly bounded above. The result follows. \square

We now consider the second sum in Equation (16).

Lemma 6.5. *Let $\tilde{\alpha}_j$ be a lift of the curve α along which the curve ζ has shift coordinate $S_j = S_j(\zeta)$. Then*

$$\frac{\partial S_j}{\partial l(\alpha)} = tw_{\mathcal{L}_t}(\zeta, \alpha) - s_\alpha(\mathcal{L}_t) \pm O(1),$$

where $s_\alpha(\mathcal{L}_t)$ is the Fenchel-Nielsen twist along α at \mathcal{L}_t as defined in Section 4.2.

Proof. Homotope the lift $\tilde{\zeta}$ to the piecewise geodesic path $\hat{\zeta}$ that runs along the successive lifts $\tilde{\alpha}_i$ and common perpendiculars π_i . The projection of $\hat{\zeta}$ to S is homotopic to ζ . Then $S_j(\zeta)$ equals the signed distance that $\hat{\zeta}$ travels along $\tilde{\alpha}_j$. We need to express this shift in a usable form.

Recall the definition of Fenchel-Nielsen twist coordinates from Section 4.2. As above, we denote the pants curves in a short marking for \mathcal{L}_t by $\alpha_1, \dots, \alpha_k$. The curve $\tilde{\alpha}_j$ forms the boundary of the lifts of two, possibly coincident, pairs of pants P_j and P_{j+1} . The projection $\hat{\alpha}_j$ of $\tilde{\alpha}_j$ to P_j is bisected by the endpoints of the two seams of P_j which join $\hat{\alpha}_j$ to each of the other two boundaries of P_j (before identification in the surface S). Likewise the projection $\hat{\alpha}'_j$ of $\tilde{\alpha}_j$ to P_{j+1} is bisected by the endpoints of exactly two seams of P_{j+1} .

The zero twist surface $\sigma_0 = \sigma_0(l_{\alpha_1}, \dots, l_{\alpha_k})$ is formed by gluing P_j to P_{j+1} along $\hat{\alpha}_j$ and $\hat{\alpha}'_j$ in such a way as to match these two pairs of points. Thus on σ_0 , the distance along $\tilde{\alpha}_j$ between incoming and outgoing perpendiculars π_j and π_{j+1} may be expressed in the form $n_j(\zeta)l(\alpha)/2 + e_j(\zeta)$, where $n_j(\zeta) \in \mathbb{Z}$ is the (signed) number of seams $\hat{\zeta}$ intersects along $\tilde{\alpha}_j$ and $e_j(\zeta)$ is an error term which allows for the possibility that π_j, π_{j+1} may not be seams of P_j and P_{j+1} , but rather common perpendiculars from $\hat{\alpha}_j$ or $\hat{\alpha}'_j$ to itself. In all cases however, $|e_j(\zeta)| < l_\sigma(\alpha_j)$ and $e_j(\zeta)$ depends only on the geometry of P_j and P_{j+1} , see [24].

Now at \mathcal{L}_t , the incoming and outgoing perpendiculars π_j and π_{j+1} are further offset by $\tau_{\alpha_j}(\mathcal{L}_t) = l_{\mathcal{L}_t}(\alpha)s_{\alpha_j}(\mathcal{L}_t)$ giving the formula

$$S_j(\zeta) = \frac{1}{2}n_j(\zeta)l_{\mathcal{L}_t}(\alpha) + e_j(\zeta) + \tau_{\alpha_j}(\mathcal{L}_t),$$

see also Section 4.2 of [24].

We can now proceed to estimate $\partial S_j / \partial l(\alpha)$. Since the partial derivatives are taken with respect to the coordinates $(l_\sigma(\alpha), \tau_\alpha(\sigma))$, the term $\partial \tau_{\alpha_j} / \partial l(\alpha)$ vanishes. To avoid an unpleasant calculation, we get rid of the term $e_j(\zeta)$ as follows. Modify $\hat{\zeta}$ to a path which still runs along the

lifts of the pants curves and their common perpendiculars, but which never goes along a perpendicular from a lift of α to itself. Specifically, let π_j be such a common perpendicular which projects to a pair of pants P one of whose boundary components is α . Let β be one of the other boundary components and let η be the perpendicular from α to β . The projection of π_j to P is homotopic, with fixed endpoints, to an arc which runs along α , then along η , then along β , back along η , finally back to the final point on α , see Figure 3. Modify $\hat{\zeta}$ by replacing

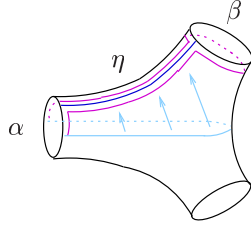


FIGURE 3. Homotoping the perpendicular.

π_j by the lift of this alternate path. Doing this in each instance gives a replacement for $\hat{\zeta}$, with respect to which one can define all quantities occurring in (16) as before. The derivation of Equation (16) in [24] will still work for this new path. Denoting the newly defined shift also by S_j , we thus have $\partial S_j / \partial l(\alpha) = n_j / 2 \pm 1$.

We claim that $n_j / 2 = tw_{\sigma_0}(\zeta, \alpha) \pm O(1)$. By definition, $\hat{\zeta}$ traverses lifts of the pants curves to \mathbb{H} in the same order as $\tilde{\zeta}$. Thus the segment of $\hat{\zeta}$ running along the lift $\tilde{\alpha} = \tilde{\alpha}_j$ to \mathbb{H} is the interval between the footpoints Q_j, Q_{j+1} of the perpendiculars π_j and π_{j+1} from $\tilde{\alpha}_{j-1}$ and $\tilde{\alpha}_{j+1}$ (the lifts of pants curves adjacent to $\tilde{\alpha}_j$) to $\tilde{\alpha}_j$. Now Q_j lies within the interval on $\tilde{\alpha}_j$ bounded by the footpoints of the perpendiculars from the two endpoints of $\tilde{\alpha}_{j-1}$ on $\partial\mathbb{H}$ to $\tilde{\alpha}_j$; and similarly for Q_{j+1} . Thus our claim follows as in [20] Lemma 3.1, see the discussion in Section 4.1. The proof of the present lemma can now be completed by applying Lemma 4.1. \square

We can now put the above results together to obtain an estimate of $\partial l(\nu) / \partial l(\alpha)$. Let $\{\pi_j\}_{j \in J}$ be the subset of perpendiculars whose projections are contained in a common pair of pants with α but are disjoint from α . Then by Lemmas 6.3(ii) and 6.4, we have

$$\sum_{j \in J} \cosh u_j \frac{\partial d_j}{\partial l(\alpha)} \stackrel{*}{\asymp} \sum_{j \in J} \frac{\partial d_j}{\partial l(\alpha)} \stackrel{*}{\asymp} \sum_{j \in J} l(\alpha).$$

In the case that we have a measured lamination ν instead of a curve ζ , we obtain by the same reasoning

$$\sum_{j \in J} \cosh u_j \frac{\partial d_j}{\partial l(\alpha)} \stackrel{*}{\asymp} \sum_{j \in J} l(\alpha) \cdot w_\nu(\pi_j),$$

where $w_\nu(\pi_j)$ is the ν -weight of the leaves of $\tilde{\nu}$ that go from $\tilde{\alpha}_j$ to $\tilde{\alpha}_{j+1}$. Let us denote

$$(17) \quad \Delta_\nu(\alpha) = \sum_{j \in J} w_\nu(\pi_j).$$

By applying Equation (15) and Lemmas 6.3 – 6.5 to Equation (16), we obtain the following:

Lemma 6.6. *Let α be a curve in a short pants decomposition $\mathcal{P}_{\mathcal{L}_t}$ of \mathcal{L}_t and let ν be a measured lamination transverse to α , with average intersection angle Θ . If α is extremely short and neither horizontal nor vertical, then using coordinates $(l(\alpha_i), \tau_{\alpha_i})$ relative to $\mathcal{P}_{\mathcal{L}_t}$, we have $\partial l(\nu)/\partial l(\alpha) = -A + B + C$, where*

$$\begin{aligned} A &\stackrel{*}{\asymp} i(\nu, \alpha) \frac{1}{l(\alpha)}, & B &\stackrel{*}{\asymp} \Delta_\nu(\alpha) l(\alpha) & \text{and} \\ C &\stackrel{*}{\asymp} i(\nu, \alpha) ((tw_{\mathcal{L}_t}(\nu, \alpha) - s_\alpha(\mathcal{L}_t)) \cos \Theta + O(1)). \end{aligned}$$

6.3. Proof of the main estimates. We are ready to prove our main results Theorems 6.1 and 6.2. If we apply Equation (13) to $\partial/\partial l(\alpha)$, then by Lemma 6.6, we obtain

$$(18) \quad 0 = \frac{\partial l(\nu_t^+)}{\partial l(\alpha)} + \frac{\partial l(\nu_t^-)}{\partial l(\alpha)} = -(A^+ + A^-) + B^+ + B^- + C^+ + C^-,$$

where

$$\begin{aligned} A^\pm &\stackrel{*}{\asymp} \frac{i(\nu_t^\pm, \alpha)}{l(\alpha)}, & B^\pm &\stackrel{*}{\asymp} \Delta_{\nu_t^\pm}(\alpha) l(\alpha) & \text{and} \\ C^\pm &\stackrel{*}{\asymp} i(\nu_t^\pm, \alpha) ((tw_{\mathcal{L}_t}(\nu^\pm, \alpha) - s_\alpha(\mathcal{L}_t)) \cos \Theta_t^\pm + O(1)). \end{aligned}$$

Since $A^+ + A^- = B^+ + B^- + C^+ + C^-$, we get

$$(19) \quad \frac{1}{l(\alpha)} \stackrel{*}{\asymp} \frac{\Delta_{\nu_t^+}(\alpha) + \Delta_{\nu_t^-}(\alpha)}{i(\nu_t^+, \alpha) + i(\nu_t^-, \alpha)} l(\alpha) + \frac{C^+ + C^-}{i(\nu_t^+, \alpha) + i(\nu_t^-, \alpha)}.$$

Notice that the term $C^+ + C^-$ simplifies: defining

$$D^\pm = C^\pm + i(\nu_t^\pm, \alpha) s_\alpha(\mathcal{L}_t) \cos \Theta_t^\pm,$$

it follows immediately from Equation (14) that $C^+ + C^- = D^+ + D^-$.

Lemma 6.7. *Let $H_t(\alpha)$ be defined as in Equation (12). Then we have*

$$H_t(\alpha) \asymp \frac{\Delta_{\nu_t^+}(\alpha) + \Delta_{\nu_t^-}(\alpha)}{i(\nu_t^+, \alpha) + i(\nu_t^-, \alpha)}.$$

Proof. The strands of ν_t^\pm which intersect pants adjacent to α but which are disjoint from α , must intersect one of the curves in \mathcal{B} . Hence, by definition of $\Delta_{\nu_t^\pm}(\alpha)$, we have

$$\frac{\Delta_{\nu_t^+}(\alpha) + \Delta_{\nu_t^-}(\alpha)}{i(\nu_t^+, \alpha) + i(\nu_t^-, \alpha)} \prec \sum_{\beta \in \mathcal{B}} \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)} \stackrel{*}{\asymp} H_t(\alpha).$$

To prove the inequality in the other direction, let $\beta \in \mathcal{B}$ be the curve that realizes the maximum in the definition of $H_t(\alpha)$. For $\nu = \nu_t^\pm$, let $\nu_{\beta\alpha}, \nu_{\beta\beta}, \nu_{\beta\gamma}$ be the collections of strands of ν that run between β and α , from β to itself, and between β and γ , respectively. (As usual, there are different possible configurations of strands in each pants, in particular $\nu_{\beta\beta}$ may be empty. The inequalities which follow are however valid in all cases.) Denote the ν -weight of these by $w(\nu_{\beta\alpha}), w(\nu_{\beta\beta}),$ and $w(\nu_{\beta\gamma})$, respectively. Then

$$\begin{aligned} H_t(\alpha) &= \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)} \stackrel{*}{\asymp} \sum_{I=+,-} \frac{w(\nu_{\beta\alpha}^I) + w(\nu_{\beta\beta}^I) + w(\nu_{\beta\gamma}^I)}{l_{q_t}(\alpha)} \\ &\prec \sum_{I=+,-} \frac{w(\nu_{\beta\beta}^I) + w(\nu_{\beta\gamma}^I)}{l_{q_t}(\alpha)} \prec \frac{\Delta_{\nu_t^+}(\alpha) + \Delta_{\nu_t^-}(\alpha)}{i(\nu_t^+, \alpha) + i(\nu_t^-, \alpha)}. \quad \square \end{aligned}$$

Proof of Theorem 6.1. Lemma 6.7, Equation (19), and the remark following gives $1/l_{\mathcal{L}_t}(\alpha) \asymp G_t + H_t l_{\mathcal{L}_t}(\alpha)$, where $H_t = H_t(\alpha)$ and

$$G_t = G_t(\alpha) = \frac{D^+ + D^-}{i(\nu_t^+, \alpha) + i(\nu_t^-, \alpha)}.$$

Hence, we must have either

$$\frac{1}{l_{\mathcal{L}_t}(\alpha)} \asymp G_t \quad \text{or} \quad \frac{1}{l_{\mathcal{L}_t}(\alpha)} \asymp H_t l_{\mathcal{L}_t}(\alpha),$$

from which we obtain

$$\frac{1}{l_{\mathcal{L}_t}(\alpha)} \asymp \max\{G_t, \sqrt{H_t}\}.$$

We simplify the expression for G_t as follows. By the discussion following Equation (15) we see that either $tw_{\mathcal{L}_t}(\nu^+, \alpha) \cos \Theta_t^+$ is positive, or $|tw_{\mathcal{L}_t}(\nu^+, \alpha) \cos \Theta_t^+| = O(1)$, and likewise for ν^- . Also note that

$tw_{\mathcal{L}_t}(\nu^+, \alpha)$ and $tw_{\mathcal{L}_t}(\nu^-, \alpha)$ are either $O(1)$ or have opposite signs, so that

$$d_\alpha(\nu^+, \nu^-) = Tw_{\mathcal{L}_t}(\nu^+, \alpha) + Tw_{\mathcal{L}_t}(\nu^-, \alpha) \pm O(1).$$

As before, let $T = t - t_\alpha$. Then by applying Equation (14), we get

$$\begin{aligned} G_t(\alpha) &= \frac{e^T |\cos \Theta_t^+| Tw_{\mathcal{L}_t}(\nu^+, \alpha) + e^{-T} |\cos \Theta_t^-| Tw_{\mathcal{L}_t}(\nu^-, \alpha)}{e^T + e^{-T}} + O(1) \\ (20) \quad &\asymp \frac{e^T |\cos \Theta_t^+| d_\alpha(\nu^+, \nu^-)}{e^T + e^{-T}} = \frac{e^{-T} |\cos \Theta_t^-| d_\alpha(\nu^+, \nu^-)}{e^T + e^{-T}}. \end{aligned}$$

This almost completes the proof, except it remains to be shown that if $1/l_{\mathcal{L}_t}(\alpha) \asymp G_t$, then

$$G_t \asymp e^{-2|t-t_\alpha|} d_\alpha(\nu^+, \nu^-).$$

By Equation (20), it is sufficient to show that there is some constant $c > 0$, independent of α , such that $|\cos \Theta_t^-| > c$ whenever $T > 0$ and $|\cos \Theta_t^+| > c$ whenever $T < 0$.

Our assumption that $1/l_{\mathcal{L}_t}(\alpha) \asymp G_t$ and the fact that $l_{\mathcal{L}_t}(\alpha)$ is sufficiently small, together with Equation (20) imply that

$$\frac{1}{l_{\mathcal{L}_t}(\alpha)} \prec^* Tw_{\mathcal{L}_t}(\nu^-, \alpha) + Tw_{\mathcal{L}_t}(\nu^+, \alpha).$$

Let $X_t = Tw_{\mathcal{L}_t}(\nu^-, \alpha)l_{\mathcal{L}_t}(\alpha)$ and $Y_t = Tw_{\mathcal{L}_t}(\nu^+, \alpha)l_{\mathcal{L}_t}(\alpha)$. The above inequality states that

$$(21) \quad X_t + Y_t \succ^* 1.$$

If $T > 0$, then by Equation (14), $|\cos \Theta_t^-| > |\cos \Theta_t^+|$ so $X_t > Y_t - O(l_{\mathcal{L}_t}(\alpha))$ by Equation (15). Thus, reducing the value of the upper bound ϵ_0 on $l_{\mathcal{L}_t}(\alpha)$ if necessary, it follows from Equation (21) that X_t is bounded below by some positive constant, and thus the same is true of $|\cos \Theta_t^-|$. The analogous statement holds for $|\cos \Theta_t^+|$ when $T < 0$. \square

Proof of Theorem 6.2. From Equation (14), $|\cos \Theta_t^\pm| \leq e^{-2|T|}$. It follows from Equation (15) that if $T \gg 0$ then

$$Tw_{\mathcal{L}_t}(\nu^+, \alpha)l_{\mathcal{L}_t}(\alpha) \prec e^{-2T}.$$

The argument for $T \ll 0$ is similar. Now suppose that $|T| = O(1)$. Since

$$Tw_{\mathcal{L}_t}(\nu^\pm, \alpha)l_{\mathcal{L}_t}(\alpha) \prec d_\alpha(\nu^+, \nu^-)l_{\mathcal{L}_t}(\alpha)$$

and since by Theorem 6.1,

$$d_\alpha(\nu^+, \nu^-)l_{\mathcal{L}_t}(\alpha) \prec e^{2|T|},$$

the result follows. \square

6.4. Estimates for horizontal and vertical short curves on \mathcal{L} .

As in Section 5.6, we need the analogue of Theorems 6.1 and 6.2 for extremely short curves α which are either horizontal or vertical. As in that section, assume α is vertical so that $i(\alpha, \nu^-) = 0$.

As before, we shall obtain the estimates by applying Equation (13) to $\frac{\partial}{\partial \tau_\alpha}, \frac{\partial}{\partial l(\alpha)}$. Since α is vertical,

$$0 = \frac{\partial l(\nu_t^+)}{\partial \tau_\alpha} + \frac{\partial l(\nu_t^-)}{\partial \tau_\alpha} = \frac{\partial l(\nu_t^+)}{\partial \tau_\alpha} = \int_\alpha \cos \theta^+ d\nu_t^+.$$

Hence Equation (14) is replaced by $\cos \Theta_t^+ = 0$. Furthermore, Equation (15) gives $|tw_{\mathcal{L}_t}(\nu^+, \alpha)| = O(1)$.

Let $m^-(\alpha)$ be the weight on α of $\nu^- = \nu_0^-$, in other words, $\nu^- = m^-(\alpha)\alpha + \eta$, where η has support disjoint from α . Then following the line of discussion in Section 6.2, it is easy to check that

$$\frac{\partial l(\nu_t^-)}{\partial l(\alpha)} = e^{-t} m^-(\alpha) + \Delta_{\nu_t^-}(\alpha) l(\alpha).$$

Hence, in place of Equation (18), we obtain

$$0 = \frac{\partial l(\nu_t^+)}{\partial l(\alpha)} + \frac{\partial l(\nu_t^-)}{\partial l(\alpha)} = -A^+ + B^+ + C^+ + e^{-t} m^-(\alpha) + B^-,$$

where A^\pm, B^\pm, C^\pm are defined as before. Since

$$C^+ \asymp^* i(\nu_t^+, \alpha) [(tw_{\mathcal{L}_t}(\nu, \alpha) - s_\alpha(\mathcal{L}_t)) \cos \Theta_t^+ + O(1)] \asymp^* i(\nu_t^+, \alpha),$$

we get

$$\frac{1}{l_{\mathcal{L}_t}(\alpha)} \asymp H_t(\alpha) l_{\mathcal{L}_t}(\alpha) + \frac{e^{-t} m^-(\alpha)}{i(\nu_t^+, \alpha)} = H_t(\alpha) l_{\mathcal{L}_t}(\alpha) + e^{-2t} \frac{m^-(\alpha)}{i(\nu^+, \alpha)}.$$

Thus we obtain:

Theorem 6.8. *Let α be a curve which is vertical on S . If α is extremely short in \mathcal{L}_t , then*

$$\frac{1}{l_{\mathcal{L}_t}(\alpha)} \asymp \max \left\{ e^{-2t} \frac{m^-(\alpha)}{i(\nu^+, \alpha)}, \sqrt{H_t(\alpha)} \right\}.$$

If α is horizontal, the estimate is the same except that the first term is replaced by $e^{2t} m^+(\alpha)/i(\nu^-, \alpha)$, where now $m^+(\alpha)$ is the weight on α of ν^+ .

Theorem 6.9. *If α is extremely short in \mathcal{L}_t , then the twist satisfies $Tw_{\mathcal{L}_t}(\nu^+, \alpha) = O(1)$ if α is vertical and $Tw_{\mathcal{L}_t}(\nu^-, \alpha) = O(1)$ if α is horizontal.*

7. COMPARING \mathcal{L}_t AND \mathcal{G}_t

In this section we prove our final results. We compare the geometry of \mathcal{L}_t and \mathcal{G}_t by looking at their respective thick-thin decompositions. Specifically, we prove that

$$(22) \quad H_t(\alpha) \asymp K_t(\alpha).$$

Combined with Theorems 5.10 and 6.1, this completes the proof of Theorems C and A. We show further in Theorem 7.10 that on corresponding thick components, the two metrics \mathcal{L}_t and \mathcal{G}_t almost coincide. Combining this with the information about twisting given in Theorems 5.11 and 6.2, we can then estimate the Teichmüller distance between \mathcal{L}_t and \mathcal{G}_t , thus completing the proof of Theorem D (Theorem 7.15).

As explained in the Introduction, the logical flow in the proof of Equation (22) is not straightforward. We first show relatively easily in Proposition 7.1 that $H_t(\alpha) \succ K_t(\alpha)$. The key point in proving the other half of Equation (22) is Proposition 7.4, which shows that the metric \mathcal{L}_t not only minimizes $l_\sigma(\nu_t^+) + l_\sigma(\nu_t^-)$, but that it also in a suitable coarse sense minimizes the contribution to the sum made by the parts of ν_t^\pm which lie in the thick part of \mathcal{L}_t . This is proved in Section 7.2. In Section 7.3 we use Proposition 7.4 to deduce that a curve that is extremely short in \mathcal{G}_t is also extremely short in \mathcal{L}_t (Proposition 7.8). This is used in proving Theorem 7.10 mentioned above, from which in Section 7.4 we are finally able to show that $H_t(\alpha) \prec K_t(\alpha)$ (Proposition 7.12).

7.1. Curves are shorter in $\mathcal{L}(\nu^+, \nu^-)$.

Proposition 7.1. *If α is extremely short in \mathcal{G}_t , then $H_t(\alpha) \succ K_t(\alpha)$. Therefore,*

$$\frac{1}{l_{\mathcal{L}_t}(\alpha)} \succ \frac{1}{l_{\mathcal{G}_t}(\alpha)}.$$

Proof. Once we show that $H_t(\alpha) \succ K_t(\alpha)$, the second statement follows from Theorems 5.10 and 6.1.

The only case of interest is when $K_t(\alpha)$ is large. Let $E_t(\alpha)$ be one of the expanding annuli $E_i(\alpha)$ around α of larger modulus, defined as in the discussion preceding Corollary 5.4. Denote the inner and outer boundary curves of $E_t(\alpha)$ by ∂_0 and ∂_1 . Let ω be an essential arc from α to itself such that $l_{q_t}(\omega) = 2d_{q_t}(\partial_0, \partial_1)$, where as usual d_{q_t} denotes distance in the q_t -metric. The annulus $E_t(\alpha)$ intersects the q_t -representative \hat{Y} of a thick component Y of (S, \mathcal{G}_t) adjacent to α . Let us first suppose that ω is contained in \hat{Y} . A small regular neighborhood of $\alpha \cup \omega$ has boundary consisting of α and two curves, ζ_1, ζ_2 , which

together with α bound a pair of pants. Therefore, either both ζ_1 and ζ_2 are contained in \mathcal{B} or one of these two curves must intersect a curve in \mathcal{B} transversely. (As in Equation (12), \mathcal{B} is the set of pants curves in a short pants decomposition $\mathcal{P}_{\mathcal{L}_t}$ which are boundaries of pants adjacent to α .)

First consider the case when $\zeta_1, \zeta_2 \in \mathcal{B}$. If either ζ_1 or ζ_2 is non-peripheral in Y , then by definition of $H_t(\alpha)$ and λ_Y ,

$$H_t(\alpha) = \max_{\beta \in \mathcal{B}} \left\{ \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)} \right\} \geq \max \left\{ \frac{l_{q_t}(\zeta_1)}{l_{q_t}(\alpha)}, \frac{l_{q_t}(\zeta_2)}{l_{q_t}(\alpha)} \right\} \geq \frac{\lambda_Y}{l_{q_t}(\alpha)}.$$

If both ζ_1, ζ_2 are peripheral in Y , then

$$(23) \quad H_t(\alpha) \succ \frac{l_{q_t}(\zeta_1) + l_{q_t}(\zeta_2) + l_{q_t}(\alpha)}{l_{q_t}(\alpha)} \asymp \frac{\lambda_Y}{l_{q_t}(\alpha)}.$$

Now consider the case when either ζ_1 or ζ_2 intersects a curve $\beta \in \mathcal{B}$ transversely. Note that for $i = 1, 2$,

$$l_{q_t}(\zeta_i) \leq l_{q_t}(\omega) + l_{q_t}(\alpha) \leq 4 \operatorname{diam}_{q_t}(\hat{Y}) \stackrel{*}{\asymp} \lambda_Y.$$

Suppose that β intersects ζ_i and that ζ_i is not peripheral in Y . Then it follows from the above inequality and the definition of λ_Y that $l_{q_t}(\zeta_i) \stackrel{*}{\asymp} \lambda_Y$. Since by Theorem 5.5 we have $l_{q_t}(\zeta_i) \stackrel{*}{\asymp} \lambda_Y l_{\mathcal{G}_t}(\zeta_i)$, we see that $l_{\mathcal{G}_t}(\zeta_i) \stackrel{*}{\asymp} 1$. Then, by the collar lemma for quadratic differentials [21], we have

$$l_{q_t}(\beta) \stackrel{*}{\asymp} l_{q_t}(\zeta_i) \stackrel{*}{\asymp} \lambda_Y.$$

The only remaining possibility to consider is when both ζ_1, ζ_2 are peripheral so that Y is a pair of pants. If β intersects ζ_i , then since β must pass through an annulus around ζ_i which has large modulus, we conclude $l_{q_t}(\beta) \succ l_{q_t}(\zeta_i)$. If β does not intersect ζ_i , every arc η of $\beta \cap \hat{Y}$ has both endpoints on the other curve ζ_j , $j \neq i$. The endpoints of η divide ζ_j into two arcs, one of which together with η forms a curve homotopic to ζ_i . Since β passes through an annulus of large modulus around ζ_j , this implies

$$l_{q_t}(\zeta_i) \leq l_{q_t}(\eta) + l_{q_t}(\zeta_j) \stackrel{*}{\prec} l_{q_t}(\beta) + l_{q_t}(\beta).$$

Either way, we have $l_{q_t}(\beta) \stackrel{*}{\asymp} l_{q_t}(\zeta_i)$ and thus $l_{q_t}(\beta) \stackrel{*}{\asymp} l_{q_t}(\zeta_1) + l_{q_t}(\zeta_2)$.

Finally, if the original arc ω was not contained in \hat{Y} , we can replace it with an arc that is contained in \hat{Y} of comparable length as in the proof of Theorem 5.9, and run the same argument. \square

7.2. Length estimates on subsurfaces. The object of this section is to prove Proposition 7.4. We begin with estimates that are necessary to analyze the contribution to the length of a lamination associated to the thick part of the surface S . Thus if (S, σ) is a hyperbolic surface and $Y \subset S$ is a subsurface of the thick part, we want to find an approximation to $l_\sigma(\nu^\pm \cap Y)$. To consider the problem in general, we consider $l_\sigma(\nu^\pm \cap Q)$ for a subsurface Q with geodesic boundary. Suppose that ζ is a geodesic that intersects Q but is not entirely contained in Q . The essential idea is that we can approximate $\zeta \cap Q$ by piecewise geodesic arcs homotopic to $\zeta \cap Q$, which alternately run along arcs perpendicular to ∂Q and parallel to ∂Q . The length of the parallel portion is determined by the twisting of ζ about the curves in ∂Q , while the portion ζ_Q perpendicular to ∂Q is defined and estimated as explained below.

Let $\underline{\alpha}$ be a collection of disjoint simple closed geodesics on (S, σ) and let Q be a totally geodesic surface which is the metric completion of a component of $S \setminus \underline{\alpha}$. (It is possible for two distinct boundary components of Q to be identified in S to a single curve $\alpha \in \underline{\alpha}$, so strictly speaking, Q is not a subsurface of S .) If η is an essential geodesic arc with endpoints on ∂Q , let η_Q be the shortest arc in Q that is freely homotopic to η , relative to ∂Q . In this case, clearly η_Q is orthogonal to ∂Q . If φ is a measured geodesic lamination whose support is entirely contained in Q , let $\varphi_Q = \varphi$. For convenience we allow the possibility that the support of φ contains components of ∂Q , remarking that this is not quite the same as the definition in [20].

Suppose ξ is a measured geodesic lamination on S . Then the intersection $\xi \cap Q$ is a union of components of ξ that are entirely contained in Q and arcs with both endpoints on ∂Q . If η is an arc of $\xi \cap Q$, let $n(\eta_Q)$ denote the transverse measure of arcs in the homotopy class $[\eta_Q]$. The *orthogonal projection* of ξ into Q is $\xi_Q = \sum n(\eta_Q)\eta_Q + \sum \varphi_Q$, where the first sum is taken over a representative η_Q from each class of arcs in $\xi \cap Q$ and the second sum is taken over all components φ of ξ that are entirely contained in Q . Define

$$l_\sigma(\xi_Q) = \sum n(\eta_Q)l_\sigma(\eta_Q) + \sum l_\sigma(\varphi_Q).$$

If all curves in ∂Q are of uniformly bounded length, then we have the following estimate of $l_\sigma(\xi \cap Q)$ in terms of ξ_Q and $Tw_\sigma(\xi, \alpha)$:

Lemma 7.2. *Suppose $l_\sigma(\alpha_j) < \ell$ for every component α_j of ∂Q . Then there exists a constant $K = K(\ell)$ such that for any measured lamination ξ on S :*

$$\left| l_\sigma(\xi \cap Q) - \left[l_\sigma(\xi_Q) + \sum l_\sigma(\alpha_j) \frac{Tw_\sigma(\xi, \alpha_j)}{2} i(\xi, \alpha_j) \right] \right| \leq K i(\xi, \partial Q),$$

where the sum is taken over all α_j that intersect ξ transversely.

For a proof, see the Appendix. The next lemma can be proved similarly, applying the same property of hyperbolic triangles. We omit the proof.

Lemma 7.3. *Suppose $l_\sigma(\alpha) < \epsilon_0$. Let A be an embedded annulus in (S, σ) such that one component of ∂A is the geodesic α and the other a hyperbolically equidistant curve of length ϵ_0 . Then there is a uniform constant K such that for any measured lamination ξ on S that intersects α transversely:*

$$\left| l_\sigma(\xi \cap A) - \left[\log \frac{1}{l_\sigma(\alpha)} + l_\sigma(\alpha) \frac{Tw_\sigma(\xi, \alpha)}{2} \right] i(\xi, \alpha) \right| \leq K i(\xi, \alpha).$$

Here, $\log[1/l_\sigma(\alpha)]$ approximates the width of A , up to a bounded additive error.

We will now apply Lemmas 7.2 and 7.3 to prove Proposition 7.4 below, which in turn is the key step to proving Theorem 7.10.

For $\rho > 0$ and a hyperbolic metric $\sigma \in \mathcal{T}(S)$, define

$$\mathcal{S}_\rho(\sigma) = \{\alpha \in \mathcal{S} : l_\sigma(\alpha) < \rho\}.$$

Proposition 7.1 implies that if α is extremely short in \mathcal{G}_t , then we can choose a constant $\epsilon < \epsilon_0$, depending only on ϵ_0 , such that if $l_{\mathcal{G}_t}(\alpha) < \epsilon$, then $l_{\mathcal{L}_t}(\alpha) < \epsilon_0$. In other words, we can choose $\epsilon < \epsilon_0$ so that

$$(24) \quad \mathcal{S}_\epsilon(\mathcal{G}_t) \subset \mathcal{S}_{\epsilon_0}(\mathcal{L}_t).$$

Now, let $Q = Q_t$ be a component of $S \setminus \mathcal{S}_\epsilon(\mathcal{G}_t)$. The metric \mathcal{L}_t naturally endows Q with the structure of hyperbolic surface with geodesic boundary, which by the above, satisfies $l_{\mathcal{L}_t}(\alpha) < \epsilon_0$ for all components α of ∂Q . Henceforth, fix a constant c that satisfies $\epsilon_0 < c < \epsilon_{\mathcal{M}}$. For $\sigma = \mathcal{G}_t, \mathcal{L}_t$, or in general, any metric that makes Q a hyperbolic surface with geodesic boundary components that are extremely short, define $C(\alpha, \sigma)$ to be the collar of α in (Q, σ) such that one component of $\partial C(\alpha)$ is (the geodesic representative of) α , and the other, the equidistant curve of length c . Because $c < \epsilon_{\mathcal{M}}$, the collars are all disjoint from one another. Let (Q_T, σ) be the metric subsurface of Q defined by:

$$(Q_T, \sigma) = (Q, \sigma) \setminus \coprod_{\alpha \in \partial Q} C(\alpha, \sigma).$$

In particular, every component of ∂Q_T has length c .

Since \mathcal{L}_t is on the line of minima, we have

$$l_{\mathcal{L}_t}(\nu_t^+) + l_{\mathcal{L}_t}(\nu_t^-) \leq l_{\mathcal{G}_t}(\nu_t^+) + l_{\mathcal{G}_t}(\nu_t^-).$$

The contribution to this inequality from Q_T is given as follows:

Proposition 7.4. *If Q is a component of $S \setminus \mathcal{S}_\epsilon(\mathcal{G}_t)$, then*

$$l_{\mathcal{L}_t}(\nu^+ \cap Q_T) + l_{\mathcal{L}_t}(\nu^- \cap Q_T) \stackrel{*}{\prec} l_{\mathcal{G}_t}(\nu^+ \cap Q_T) + l_{\mathcal{G}_t}(\nu^- \cap Q_T).$$

To prove Proposition 7.4, we need the following two lemmas.

Lemma 7.5. *Suppose (Q, σ) is a hyperbolic surface with geodesic boundary such that $l_\sigma(\alpha) < \epsilon_0$ for all $\alpha \in \partial Q$. Then for any measured geodesic lamination ξ on (S, σ) , we have*

$$l_\sigma(\xi \cap Q_T) \stackrel{*}{\asymp} l_\sigma(\xi_Q \cap Q_T).$$

Proof. We consider $l_\sigma(\xi \cap Q_T) = l_\sigma(\xi \cap Q) - l_\sigma(\xi \cap \coprod C(\alpha))$ and apply Lemmas 7.2, 7.3 to the right-hand side. The proof is completed by observing that

$$l_\sigma(\xi_Q \cap Q_T) = l_\sigma(\xi_Q) - \sum_{\alpha \in \partial Q} \log \frac{1}{l_\sigma(\alpha)} i(\xi, \alpha) + O(i(\xi, \partial Q))$$

and that $l_\sigma(\xi \cap Q_T) \stackrel{*}{\asymp} i(\xi, \partial Q_T) = i(\xi, \partial Q)$, due to the fact that every component of ∂Q_T has an annular neighborhood of definite width. \square

Lemma 7.6. *Suppose that (Q, σ) and (Q, σ') are two hyperbolic surfaces with geodesic boundary whose boundary components are all extremely short. Suppose that there is a short pants decomposition of (Q, σ) with respect to which the Fenchel-Nielsen coordinates for (Q, σ) and (Q, σ') agree, except possibly for the lengths and twists corresponding to components of ∂Q . Then for any simple closed curve or arc η with endpoints on ∂Q ,*

$$l_\sigma(\eta_Q \cap Q_T) \stackrel{*}{\asymp} l_{\sigma'}(\eta_Q \cap Q_T).$$

Proof. This is essentially the same as a discussion in Minsky [20] page 283. The idea is that there is a K -bilipschitz homeomorphism $(Q_T, \sigma) \rightarrow (Q_T, \sigma')$ with constant K depending only on ϵ_0 . To see this, cut Q along the pants curves into pairs of pants and further cut each pair of pants into hexagons. Corresponding hexagons in the two surfaces have the same side lengths, except those whose edges form part of ∂Q . Now truncate those hexagons which have an edge on ∂Q by cutting off the collar round ∂Q in such a way that the boundary of the truncated hexagon is the corresponding component of ∂Q_T . By our construction, the non-geodesic edges of the hexagons in the two surfaces are both equidistant curves of the same length $c/2$.

We define the required map piecewise from each possibly truncated hexagon in (Q_T, σ) to the corresponding one in (Q_T, σ') . Since all the

Fenchel-Nielsen coordinates agree in the interior of Q_T , we only have to see that there is a bilipschitz map between the truncated parts of two hexagons H and H' with alternate sidelengths l_1, l_2, l_3 and l'_1, l_2, l_3 coming from the pants curves, where $l_1, l'_1 < c/2$. Since l_2, l_3 are uniformly bounded above and below, the distance between the corresponding sides in both hexagons is also uniformly bounded above and below, see the proof of Lemma 3.3. The distance between the side of length l_1 and the equidistant curve of length $c/2$ is equal to $\log[c/2l_1]$, up to a bounded additive error. Hence by Lemma 3.3, the distances between the sides of lengths l_2, l_3 and the equidistant curve of length $c/2$ are bounded above, while they are bounded below by choice of c .

It is now easy to define a bilipschitz homeomorphism between the truncations of H and H' . For example, fix a point O whose distance from all sides of H is uniformly bounded above and below and divide H into six triangles by joining O to the vertices of H . Note that if we are given two hyperbolic triangles whose side lengths are uniformly bounded above and below, we can map the three sides linearly to each other and then extend to a uniformly bilipschitz map on the interiors. We can do the same even when one side is an equidistant curve rather than a geodesic. Now define the required map from H to H' triangle by triangle making it agree on the edges joining O to the vertices of H . It is clear that the resulting bound on K depends only on the initial upper bound on the lengths of the pants curves. Note also that $K \rightarrow 1$ as $\epsilon_0 \rightarrow 0$. \square

We may assume that in the definition of the truncated surfaces Q_T , the constants ϵ_0 and c are chosen small enough that any non-peripheral simple geodesic loop contained in (Q, σ) is completely contained in Q_T . In particular, if \mathcal{P}_σ is a short pants system for σ and if $\beta \in \mathcal{P}_\sigma \setminus \partial Q$ is contained in Q_T , then so is its dual δ_β . Let M_σ be the short marking of σ associated to \mathcal{P}_σ . We call the subset of M_σ thus defined, the *restriction* $M_\sigma|_Q$ of M_σ to Q [17]. Equivalently, $M_\sigma|_Q$ is the set of curves in M_σ that are completely contained in Q and are non-peripheral in Q . If Q is a pair of pants then $M_\sigma|_Q$ is empty.

Proof of Proposition 7.4. Where convenient, we drop the subscript t . Since Q is a component of $S \setminus \mathcal{S}_\epsilon(\mathcal{G}_t)$, the curves in ∂Q are included in the set of pants curves in both $M_\mathcal{L}$ and $M_\mathcal{G}$. Define a new metric $\tau = \tau_t$ on S interpolating \mathcal{G}_t and \mathcal{L}_t as follows. Let X be the metric completion of $S \setminus Q$. First we choose a new pants system \mathcal{P}_τ for S . The system \mathcal{P}_τ contains all the curves in ∂Q , in the interior of Q it consists of the pants curves in $M_\mathcal{G}|_Q$, while in the interior of X it consists of the pants curves in $M_\mathcal{L}|_X$. We define τ_t by specifying its Fenchel-Nielsen coordinates

with respect to \mathcal{P}_τ . The metric τ_t will have the same Fenchel-Nielsen coordinates associated to the pants curves in $M_{\mathcal{G}}|_Q$ as \mathcal{G}_t and the same Fenchel-Nielsen coordinates associated to the curves in $M_{\mathcal{L}}|_Q \cup \partial Q$ as \mathcal{L}_t .

Since \mathcal{L} is on the line of minima we have:

$$(25) \quad l_{\mathcal{L}}(\nu_t^+) + l_{\mathcal{L}}(\nu_t^-) \leq l_\tau(\nu_t^+) + l_\tau(\nu_t^-).$$

Let us estimate both sides of this inequality. Applying Lemma 7.2 to $\nu = \nu_t^\pm$, we obtain:

$$l_\tau(\nu \cap X) = l_\tau(\nu_X) + \frac{1}{2} \sum_{\alpha \in \partial Q} l_\tau(\alpha) i(\alpha, \nu) Tw_\tau(\nu, \alpha) + O(i(\nu, \partial Q)),$$

$$l_{\mathcal{L}}(\nu \cap X) = l_{\mathcal{L}}(\nu_X) + \frac{1}{2} \sum_{\alpha \in \partial Q} l_{\mathcal{L}}(\alpha) i(\alpha, \nu) Tw_{\mathcal{L}}(\nu, \alpha) + O(i(\nu, \partial Q)).$$

By construction, $l_\tau(\nu_X) = l_{\mathcal{L}}(\nu_X)$.

$$\begin{aligned} |l_\tau(\nu \cap X) - l_{\mathcal{L}}(\nu \cap X)| &\leq \\ &\leq \frac{1}{2} \sum_{\alpha \in \partial Q} l_\tau(\alpha) i(\nu, \alpha) |Tw_\tau(\nu, \alpha) - Tw_{\mathcal{L}}(\nu, \alpha)| + O(i(\nu, \partial Q)). \end{aligned}$$

By construction, the Fenchel-Nielsen twist coordinates for τ and \mathcal{L} on any component α of ∂Q coincide. Therefore, by Lemma 4.1, we have $|Tw_\tau(\nu, \alpha) - Tw_{\mathcal{L}}(\nu, \alpha)| \leq 4$. Thus

$$|l_\tau(\nu \cap X) - l_{\mathcal{L}}(\nu \cap X)| \leq 2 \sum_{\alpha \in \partial Q} l_\tau(\alpha) i(\nu, \alpha) + O(i(\nu, \partial Q)).$$

Substituting this into Equation (25) and noting that we are working under the assumption that all components α of ∂Q are extremely short in τ , we obtain:

$$l_{\mathcal{L}}(\nu_t^+ \cap Q) + l_{\mathcal{L}}(\nu_t^- \cap Q) < l_\tau(\nu_t^+ \cap Q) + l_\tau(\nu_t^- \cap Q) + O(l_{q_t}(\partial Q)).$$

Since every component of ∂Q is extremely short in τ , the collar lemma implies that $i(\nu, \partial Q) \stackrel{*}{\prec} l_\tau(\nu \cap Q)$ so we may replace the last approximation by

$$l_{\mathcal{L}}(\nu_t^+ \cap Q) + l_{\mathcal{L}}(\nu_t^- \cap Q) \stackrel{*}{\prec} l_\tau(\nu_t^+ \cap Q) + l_\tau(\nu_t^- \cap Q).$$

Since for $\sigma = \mathcal{L}, \tau$ and $\nu = \nu_t^\pm$, we have by Lemma 7.5

$$\begin{aligned} l_\sigma(\nu \cap Q) &= l_\sigma(\nu \cap Q_T) + l_\sigma(\nu \cap (Q \setminus Q_T)) \\ &\stackrel{*}{\asymp} l_\sigma(\nu_Q \cap Q_T) + l_\sigma(\nu \cap (Q \setminus Q_T)), \end{aligned}$$

we can apply Lemma 7.3 to subtract the contribution of the collars forming $Q \setminus Q_T$ from both sides to obtain:

$$l_{\mathcal{L}}(\nu_Q^+ \cap Q_T) + l_{\mathcal{L}}(\nu_Q^- \cap Q_T) \stackrel{*}{\prec} l_{\tau}(\nu_Q^+ \cap Q_T) + l_{\tau}(\nu_Q^- \cap Q_T).$$

To complete the proof, we apply Lemmas 7.5 and 7.6:

$$\begin{aligned} l_{\tau}(\nu_Q^+ \cap Q_T) + l_{\tau}(\nu_Q^- \cap Q_T) &\stackrel{*}{\asymp} l_g(\nu_Q^+ \cap Q_T) + l_g(\nu_Q^- \cap Q_T) \\ &\stackrel{*}{\asymp} l_g(\nu_t^+ \cap Q_T) + l_g(\nu_t^- \cap Q_T). \end{aligned} \quad \square$$

7.3. Correspondence between thick components. This section contains the meat of our comparison between the geometries of \mathcal{L}_t and \mathcal{G}_t . We show that (Corollary 7.9) the sets of short curves on \mathcal{L}_t and \mathcal{G}_t coincide. Generalizing Theorem 3.8, we prove (Theorem 7.10 and Corollary 7.11) that the geometries of the thick parts of \mathcal{L}_t and \mathcal{G}_t are close. As in that proof, our strategy is to use short markings to estimate lengths. The main point is to use Proposition 7.4 as a substitute for the length minimization property of \mathcal{L}_t .

We need the following result which generalizes Proposition 3.6 to thick components.

Proposition 7.7. *Assume that $l_{\sigma}(\alpha) < \epsilon_0$ for every component α of ∂Q . Let $\rho > 0$ and suppose that $l_{\sigma}(\zeta) \geq \rho$ for every non-peripheral simple closed curve ζ in Q . Then for any simple closed geodesic γ on (S, σ) ,*

$$(26) \quad l_{\sigma}(\gamma \cap Q_T) \stackrel{*}{\asymp} i(M_{\sigma}|_Q, \gamma),$$

where the multiplicative constants depend only on ρ .

Proof. By Lemma 7.5 it is sufficient to prove that

$$l_{\sigma}(\gamma_Q \cap Q_T) \stackrel{*}{\asymp} i(M_{\sigma}|_Q, \gamma).$$

We modify the argument in [19] Lemma 4.7.

Notice that since Q_T is ρ -thick, it follows from Corollary 3.4 that the lengths of all the curves in $M_{\sigma}|_Q$ are bounded above. Cutting Q_T along the curves in $M_{\sigma}|_Q$, we obtain a collection of convex polygons $\{D_i\}$, together with annuli $\{A_j\}$, where one boundary component $\partial_0 A_j$ is a component of ∂Q_T , while the other component $\partial_1 A_j$ is made up of arcs in $M_{\sigma}|_Q$.

Since the total length of curves in $M_{\sigma}|_Q$ is uniformly bounded above, the length of ∂D_i is uniformly bounded above, and therefore, D_i has uniformly bounded diameter. We claim that the annuli A_j also have uniformly bounded diameter. Since the length of $\partial_0 A_j$ is bounded below by ϵ_0 , an area argument shows that the distance between $\partial_1 A_j$ and $\partial_0 A_j$ is uniformly bounded above. Since, furthermore, the lengths

of $\partial_0 A_j$ and $\partial_1 A_j$ are uniformly bounded above, it follows that A_j has uniformly bounded diameter, as claimed. Setting

$$D = \max\{\text{diam} D_i, \text{diam} A_j\}$$

gives the upper bound

$$l_\sigma(\gamma_Q \cap Q_T) \leq i(\gamma, M_\sigma|_Q) \cdot D.$$

Since the lengths of all the curves in $M_\sigma|_Q$ are bounded above, by the collar lemma, there is an embedded collar of definite radius d around every curve in $M_\sigma|_Q$. Therefore, if γ crosses $\beta \in M_\sigma|_Q$, then $l_\sigma(\gamma) > d \cdot i(\gamma, \beta)$. Let k be the number of pants curves in Q . Since there must be some $\beta \in M_\sigma|_Q$ such that $i(\gamma, \beta) \geq i(\gamma, M_\sigma|_Q)/(2k)$, we have

$$l_\sigma(\gamma_Q \cap Q_T) > \frac{d}{2k} \cdot i(\gamma, M_\sigma|_Q),$$

giving the desired lower bound. \square

Applying Proposition 7.7, we can now deduce from Proposition 7.4 that a non-peripheral curve in Q cannot be too short in \mathcal{L}_t :

Proposition 7.8. *Let Q be a component of $S \setminus \mathcal{S}_\epsilon(\mathcal{G}_t)$ where ϵ is chosen as in Equation (24). Then for any non-peripheral simple closed curve ζ in Q , we have $l_{\mathcal{L}_t}(\zeta) \stackrel{*}{\succ} 1$.*

Proof. First, we claim that

$$(27) \quad l_{\mathcal{G}_t}(\nu_t^+ \cap Q_T) + l_{\mathcal{G}_t}(\nu_t^- \cap Q_T) \stackrel{*}{\asymp} \lambda_Q.$$

To see this, let $M_{\mathcal{G}_t}$ be a short marking for \mathcal{G}_t and let $M_{\mathcal{G}_t}|_Q$ denote its restriction to Q . By Proposition 7.7, we have

$$\begin{aligned} l_{\mathcal{G}_t}(\nu_t^+ \cap Q_T) + l_{\mathcal{G}_t}(\nu_t^- \cap Q_T) &\stackrel{*}{\asymp} i(M_{\mathcal{G}_t}|_Q, \nu_t^+) + i(M_{\mathcal{G}_t}|_Q, \nu_t^-) \\ &\stackrel{*}{\asymp} l_{q_t}(M_{\mathcal{G}_t}|_Q). \end{aligned}$$

On the other hand, $l_{\mathcal{G}_t}(M_{\mathcal{G}_t}|_Q) \stackrel{*}{\asymp} 1$. Hence by Theorem 5.5

$$l_{q_t}(M_{\mathcal{G}_t}|_Q) \stackrel{*}{\asymp} \lambda_Q$$

and the claim is proved.

Now if ζ is a non-peripheral simple closed curve in Q with $l_{\mathcal{L}_t}(\zeta) < \epsilon_0$, then consideration of the collar about ζ gives the estimate

$$l_{\mathcal{L}_t}(\nu \cap Q_T) \stackrel{*}{\succ} i(\nu, \zeta) \log \frac{1}{l_{\mathcal{L}_t}(\zeta)}$$

for any $\nu \in \mathcal{ML}(S)$, so in particular,

$$l_{\mathcal{L}_t}(\nu_t^+ \cap Q_T) + l_{\mathcal{L}_t}(\nu_t^- \cap Q_T) \stackrel{*}{\succ} l_{q_t}(\zeta) \log \frac{1}{l_{\mathcal{L}_t}(\zeta)}.$$

Proposition 7.4 and Equation (27) give

$$\lambda_Q \stackrel{*}{\asymp} l_q(\zeta) \log \frac{1}{l_{\mathcal{L}_t}(\zeta)}.$$

From the definition of λ_Q we have $\lambda_Q/l_q(\zeta) \leq 1$, so that $l_{\mathcal{L}_t}(\zeta) \stackrel{*}{\asymp} 1$. \square

Proposition 7.8 and Proposition 7.1 together prove Theorem A of the Introduction, that the sets of extremely short curves on \mathcal{L}_t and \mathcal{G}_t coincide. More precisely, we can reformulate Proposition 7.8 as:

Corollary 7.9. *Let ϵ be as in Equation (24). Then there exists $\epsilon' > 0$ such that $\mathcal{S}_{\epsilon'}(\mathcal{L}_t) \subset \mathcal{S}_{\epsilon}(\mathcal{G}_t)$.*

It is also now easy to complete the proof of our main comparison between the thick parts of \mathcal{L}_t and \mathcal{G}_t :

Theorem 7.10. *Let Q be a component of $S \setminus \mathcal{S}_{\epsilon}(\mathcal{G}_t)$ which is not a pair of pants, and let $M_{\mathcal{L}_t}$ be a short marking for \mathcal{L}_t . Then*

$$l_{\mathcal{G}_t}(M_{\mathcal{L}_t}|_Q) \stackrel{*}{\asymp} 1.$$

Proof. By Theorem 5.5, we have

$$\begin{aligned} l_{\mathcal{G}_t}(M_{\mathcal{L}_t}|_Q) &\stackrel{*}{\asymp} \frac{1}{\lambda_Q} l_{q_t}(M_{\mathcal{L}_t}|_Q) \\ &\stackrel{*}{\asymp} \frac{1}{\lambda_Q} [i(M_{\mathcal{L}_t}|_Q, \nu^+) + i(M_{\mathcal{L}_t}|_Q, \nu^-)]. \end{aligned}$$

By Proposition 7.8, there is a constant $\rho = \rho(\epsilon_0)$ depending only on ϵ_0 such that $l_{\mathcal{L}_t}(\zeta) > \rho(\epsilon_0)$ for every non-peripheral curve ζ in Q . Therefore, we can apply Proposition 7.7 to get

$$i(M_{\mathcal{L}_t}|_Q, \nu_t^+) + i(M_{\mathcal{L}_t}|_Q, \nu_t^-) \stackrel{*}{\asymp} l_{\mathcal{L}_t}(\nu_t^+ \cap Q_T) + l_{\mathcal{L}_t}(\nu_t^- \cap Q_T).$$

Since the lower bound $l_{\mathcal{G}_t}(M_{\mathcal{L}_t}|_Q) \stackrel{*}{\asymp} 1$ is trivial, the result now follows from Proposition 7.4 and Equation (27). \square

Equivalently, we can formulate Theorem 7.10 in terms of the surface Q_0 obtained from Q by replacing every boundary component with a puncture. Let $\mathcal{L}_t|_{Q_0}$ and $\mathcal{G}_t|_{Q_0}$ be respectively, the surface Q_0 equipped with the metrics obtained from \mathcal{L}_t and \mathcal{G}_t by pinching the curves in ∂Q but otherwise leaving the metric unchanged. In other words, in the notation of the product region theorem in Section 4.4, the collection of curves in ∂Q is \mathcal{A} and the metrics on Q_0 are defined by $\Pi_0(\mathcal{L}_t)$ and $\Pi_0(\mathcal{G}_t)$, respectively, restricted to the component Q_0 of $S_{\mathcal{A}}$. Then we have:

Corollary 7.11. *Let Q be a component of $S \setminus \mathcal{S}_\epsilon(\mathcal{G}_t)$. Then*

$$d_{T(Q_0)}(\mathcal{L}_t|_{Q_0}, \mathcal{G}_t|_{Q_0}) = O(1).$$

Proof. The boundary components of both (Q, \mathcal{L}_t) and (Q, \mathcal{G}_t) are extremely short. In this case, it was shown in [20] (see the proof of Lemma 7.6) that (Q, \mathcal{L}_t) and (Q, \mathcal{G}_t) can be embedded into $(Q_0, \mathcal{L}_t|_{Q_0})$ and $(Q_0, \mathcal{G}_t|_{Q_0})$ respectively, by a K -quasi-conformal map, where K depends only on ϵ_0 . Since simple curves do not penetrate the thin part of Q_0 , the restriction $M_{\mathcal{L}_t}|_Q$ is a short marking for $(Q_0, \mathcal{L}_t|_{Q_0})$. By Theorem 7.10, we have $l_{\mathcal{G}_t}(M_{\mathcal{L}_t}|_Q) \stackrel{*}{\asymp} 1$. The result now follows as in the proof of Theorem 3.8. \square

7.4. Comparison of lengths of short curves. Theorem 7.10 allows us to complete the proof of Equation (22):

Proposition 7.12. *Let α be an extremely short curve on \mathcal{L}_t . Then*

$$H_t(\alpha) \prec K_t(\alpha).$$

Proof. With \mathcal{B} as in Equation (12), let $\beta \in \mathcal{B}$ be the curve that has the largest q_t -length, so that

$$H_t(\alpha) \stackrel{*}{\asymp} \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)}.$$

Since $\mathcal{S}_\epsilon(\mathcal{G}_t) \subset \mathcal{S}_{\epsilon_0}(\mathcal{L}_t)$, it follows that the curves in $\mathcal{P}_{\mathcal{L}_t}$ are disjoint from $\mathcal{S}_\epsilon(\mathcal{G}_t)$. Thus, α and β are contained in the closure of a common component Q of $S \setminus \mathcal{S}_\epsilon(\mathcal{G}_t)$.

Suppose first that β is not peripheral in Q . Then $\beta \in M_{\mathcal{L}_t}|_Q$, so that by Theorem 5.5 and 7.10,

$$l_{q_t}(\beta) \stackrel{*}{\asymp} \lambda_Q l_{\mathcal{G}_t}(\beta) \stackrel{*}{\asymp} \lambda_Q.$$

If in addition, α is not peripheral, then $l_{q_t}(\alpha) \stackrel{*}{\asymp} \lambda_Q$ so that $H_t(\alpha) \stackrel{*}{\asymp} 1$ and the desired inequality holds trivially. If α is peripheral, then

$$H_t(\alpha) \stackrel{*}{\asymp} \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)} \stackrel{*}{\asymp} \frac{\lambda_Q}{l_{q_t}(\alpha)} \leq K_t(\alpha).$$

Now suppose that β is peripheral in Q . If Q is a pair of pants, then the desired inequality follows from the definition of $H_t(\alpha)$ and $K_t(\alpha)$. If Q is not a pair of pants, then since the component of $Q \setminus M_{\mathcal{G}_t}|_Q$ containing β is an annulus whose one boundary component is β and the other a finite (at most 4) union of arcs coming from curves $\cup \gamma_i$ in $M_{\mathcal{G}_t}|_Q$, again by Theorem 5.5 we obtain

$$l_{q_t}(\beta) \leq \sum l_{q_t}(\gamma_i) \stackrel{*}{\asymp} \sum \lambda_Q l_{\mathcal{G}_t}(\gamma_i) \stackrel{*}{\asymp} \lambda_Q,$$

from which the result follows as before. \square

Theorem C now follows immediately from Theorem 6.1, Proposition 7.1, and Proposition 7.12, completing our comparison between short curves on \mathcal{G}_t and \mathcal{L}_t .

Theorem 7.13. *Let α be any curve on S which is neither vertical nor horizontal. If α is extremely short in \mathcal{L}_t , then*

$$\frac{1}{l_{\mathcal{L}_t}(\alpha)} \asymp \max \left\{ D_t(\alpha), \sqrt{K_t(\alpha)} \right\}.$$

In case α is vertical or horizontal, we have

Theorem 7.14. *If α is vertical, then*

$$\frac{1}{l_{\mathcal{L}_t}(\alpha)} \asymp \max \left\{ e^{-2t} \text{Mod } F_0(\alpha), \sqrt{K_t(\alpha)} \right\}.$$

If α is horizontal, then the estimate is the same except that the first term is replaced by $e^{2t} \text{Mod } F_0(\alpha)$.

Proof. There are multiplicative constants depending only on the fixed laminations ν^\pm such that

$$\frac{m^\mp(\alpha)}{i(\nu^\pm, \alpha)} \asymp^* \text{Mod } F_0(\alpha)$$

(see Theorems 6.8 and 5.12) holds independently of α in a tautological way, due to the fact that the total number of vertical (or horizontal) curves is finite; it is bounded above by $-\chi(S)$. The proofs of Proposition 7.1 and 7.12 go through in this case, so that $H_t(\alpha) \asymp K_t(\alpha)$. Hence the estimate follows from Theorem 6.8. \square

7.5. Teichmüller distance. With the preceding collection of results at our disposal, Theorem D of the Introduction becomes an easy application of Minsky's product region theorem 4.5.

Theorem 7.15. *The Teichmüller distance between \mathcal{G}_t and \mathcal{L}_t is given by*

$$d_{\mathcal{T}(S)}(\mathcal{G}_t, \mathcal{L}_t) = \max_{\alpha \in \mathcal{S}_\epsilon(\mathcal{G}_t)} \frac{1}{2} \left| \log \frac{l_{\mathcal{G}_t}(\alpha)}{l_{\mathcal{L}_t}(\alpha)} \right| \pm O(1).$$

Proof. As noted before, $l_{\mathcal{L}_t}(\alpha) \leq \epsilon_0$ for every $\alpha \in \mathcal{S}_\epsilon(\mathcal{G}_t)$. To simplify notation, let $\mathcal{E}_t = \mathcal{S}_\epsilon(\mathcal{G}_t)$. By Theorem 4.5, we have

$$\begin{aligned} d_{\mathcal{T}(S)}(\mathcal{G}_t, \mathcal{L}_t) &= \\ &= \max_{\alpha \in \mathcal{E}_t} \{ d_{\mathcal{T}(S_{\mathcal{E}_t})}(\Pi_0(\mathcal{G}_t), \Pi_0(\mathcal{L}_t)), d_{\mathbb{H}_\alpha}(\Pi_\alpha(\mathcal{G}_t), \Pi_\alpha(\mathcal{L}_t)) \} \pm O(1), \end{aligned}$$

where $S_{\mathcal{E}_t}$ is the surface obtained from S by removing \mathcal{E}_t and replacing the resulting boundary components by punctures and Π_0, Π_α are defined as in Section 4.4. From Corollary 7.11 we deduce that

$$d_{\mathcal{T}(S_{\mathcal{E}})}(\Pi_0(\mathcal{G}_t), \Pi_0(\mathcal{L}_t)) = O(1).$$

If $t - t_\alpha > 0$, then by Theorem 5.11 or 5.13 we have

$$Tw_{\mathcal{G}_t}(\nu^+, \alpha)l_{\mathcal{G}_t}(\alpha) = O(1)$$

and by Theorem 6.2 or 6.9 we have

$$Tw_{\mathcal{L}_t}(\nu^+, \alpha)l_{\mathcal{L}_t}(\alpha) = O(1).$$

Applying Corollary 4.7 to \mathcal{G}_t and \mathcal{L}_t and the lamination ν^+ , we find

$$\exp 2d_{\mathbb{H}_\alpha}(\Pi_\alpha(\mathcal{G}_t), \Pi_\alpha(\mathcal{L}_t)) \stackrel{*}{\asymp} l_{\mathcal{G}_t}(\alpha)/l_{\mathcal{L}_t}(\alpha).$$

If $t - t_\alpha < 0$, the same result holds by applying a similar argument with ν^- . \square

7.6. Examples. The combinatorial nature of our length estimates allows us to use Theorem 7.15 to construct examples in which \mathcal{L} and \mathcal{G} have a variety of different relative behaviors. As a special case, if S is a once-punctured torus or four-times-punctured sphere, every thick component must be a pair of pants. In this case, $K_t(\alpha)$ is bounded and therefore, a curve gets short only if $d_\alpha(\nu^+, \nu^-)$ is large:

Corollary 7.16. *If S is a once-punctured torus or a four-times punctured sphere, then for any measured laminations ν^+, ν^- , the associated Teichmüller geodesic and line of minima satisfies*

$$d_{\mathcal{T}(S)}(\mathcal{G}_t, \mathcal{L}_t) = O(1).$$

On surfaces of higher genus, it is possible to have ν^+, ν^- and α such that $d_\alpha(\nu^+, \nu^-)$ is bounded while $K_t(\alpha)$ is arbitrarily large. We can construct a simple example as follows. Take two Euclidean squares each of area $1/2$ and cut open a slit of length ε at each of their centers. Although it is not necessary, for concreteness we can assume that in both squares, the slit is parallel to a pair of sides. Foliate each square

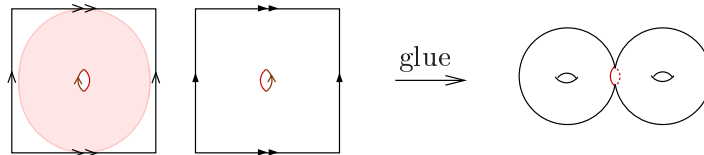


FIGURE 4. Glue two slit tori along slit.

by the two mutually orthogonal foliations that both make angle $\pi/4$

with the slit and for each, take the transverse measure induced by the Euclidean metric. Identify pairs of sides in each square to obtain two one-holed tori T_1, T_2 and glue T_1, T_2 along their boundaries, as shown in the figure, to obtain a genus two surface S with waist curve α . The two foliations match along α and specifying one to be the vertical foliation defines a quadratic differential $q = q_0$ on S . Let \mathcal{G} and \mathcal{L} be respectively, the Teichmüller geodesic and the line of minima defined by the vertical and horizontal foliations ν^- and ν^+ of q_0 . In this example, the foliations are rational, but it is easy to see that varying the initial angle of the slit gives more general foliations.

Let q_t be the associated family of quadratic differentials. Note that at time $t = 0$ the curve α is balanced. The q_t -geodesic representative of α is unique and the flat annulus corresponding to α is degenerate. Thus, by Proposition 5.8, we have $d_\alpha(\nu^+, \nu^-) = O(1)$. But since $\lambda_{T_1} = \lambda_{T_2} = 1/\sqrt{2}$ at $t = 0$, we have $K_0(\alpha) \asymp 1/\varepsilon$ (the shaded region indicates a maximal expanding annulus). Assuming ε is very small, Theorems 5.10 and 6.1 give

$$\frac{1}{l_{\mathcal{G}_0}(\alpha)} \asymp \log \frac{1}{\varepsilon} \quad \text{and} \quad \frac{1}{l_{\mathcal{L}_0}(\alpha)} \asymp \frac{1}{\sqrt{\varepsilon}}.$$

Thus by Theorem 7.15,

$$d_{\mathcal{T}(S)}(\mathcal{G}_0, \mathcal{L}_0) \succ \frac{1}{2} \log \frac{l_{\mathcal{G}_0}(\alpha)}{l_{\mathcal{L}_0}(\alpha)} \asymp \log \frac{1}{\sqrt{\varepsilon} \log[1/\varepsilon]}.$$

In fact, because for any two hyperbolic metrics σ, τ we have [28]

$$d_{\mathcal{T}(S)}(\sigma, \tau) \geq \frac{1}{2} \log \sup_{\zeta \in S} \frac{l_\sigma(\zeta)}{l_\tau(\zeta)}$$

and because the length of α along \mathcal{G} is (coarsely) shortest at the balance time $t_\alpha = 0$ [21], the following stronger inequality holds:

$$\inf_{t \in \mathbb{R}} d_{\mathcal{T}(S)}(\mathcal{G}_t, \mathcal{L}_0) \geq \inf_{t \in \mathbb{R}} \frac{1}{2} \log \frac{l_{\mathcal{G}_t}(\alpha)}{l_{\mathcal{L}_0}(\alpha)} \succ \frac{1}{2} \log \frac{l_{\mathcal{G}_0}(\alpha)}{l_{\mathcal{L}_0}(\alpha)}.$$

Taking ε small enough we can ensure that \mathcal{L}_0 is as far as we like from any point on \mathcal{G} . This example can be easily extended to any surface of large complexity, by which we mean a surface whose genus g and number of punctures p satisfies $3g - 4 + p \geq 1$. In summary,

Corollary 7.17. *If S is a surface of large complexity, then given any $n > 0$, there are measured laminations $\nu^+(n), \nu^-(n)$ on S which depend on n , such that for the associated Teichmüller geodesic $\mathcal{G}(n)$ and line of minima $\mathcal{L}(n)$,*

$$\inf_{t \in \mathbb{R}} d_{\mathcal{T}(S)}(\mathcal{G}_t(n), \mathcal{L}_0(n)) > n.$$

It is also possible to construct examples for any such surface where the two measured laminations are fixed and the associated Teichmüller geodesic and line of minima satisfy $d_{\mathcal{T}(S)}(\mathcal{G}_{t_n}, \mathcal{L}_{t_n}) > n$ for a sequence of times $t_n \rightarrow \infty$ as $n \rightarrow \infty$. This however, is beyond the scope of this paper.

8. APPENDIX

We give proofs of the length estimates that were deferred in previous sections.

Proof of Lemma 3.3. Let H be one of the two right-angled hexagons obtained by cutting P along its three seams. Let $l_i = l(\alpha_i)/2$ and d_i be the length of the seams, labeled as shown in Figure 5. By the cosine

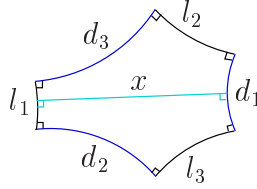


FIGURE 5. Half a pair of pants.

formula for right-angled hexagons, we have

$$\cosh d_3 = \frac{\cosh l_3 + \cosh l_1 \cosh l_2}{\sinh l_1 \sinh l_2}.$$

By hypothesis, $l_i < L/2$, so $\sinh l_i \stackrel{*}{\asymp} l_i$ and $\cosh l_i \stackrel{*}{\asymp} 1$, where the multiplicative constants involved depend only on L . Therefore,

$$\cosh d_3 \stackrel{*}{\asymp} \frac{1}{l_1 l_2}.$$

It follows that $d_3 = \log[1/l_1] + \log[1/l_2] \pm O(1)$, where the bound on the additive error depends only on L .

Now we estimate the length of the perpendicular from α_i to itself. Let x be the length of the perpendicular as in Figure 5. By the formula for right-angled pentagons, we have

$$\cosh x = \sinh l_2 \sinh d_3.$$

Since $\sinh l_2 \stackrel{*}{\asymp} l_2$ and by the above, $\sinh d_3 \stackrel{*}{\asymp} 1/[l_1 l_2]$, it follows that $\cosh x \stackrel{*}{\asymp} 1/l_1$. Hence, $x = \log[1/l_1] \pm O(1)$. Since P is made of two isometric copies of H , we obtain the desired estimate. \square

To prove Lemma 6.3, in addition to the standard hexagon and pentagon formulae (see for example [1] or [24]), we need the following expression for derivatives of side-lengths derived in [24] Proposition 2.3:

Lemma 8.1. *Let H be a planar right-angled hexagon with sides labeled $i = 1, \dots, 6$ in cyclic order about ∂H . Let l_i denote the length of side i and for $n \bmod 6$, let $p_{n,n+3}$ be the perpendicular distance from side n to side $n+3$. Letting $'$ denote derivative with respect to some variable x , we have*

$$(28) \quad (\cosh p_{n,n+3})l'_n = l'_{n+3} - (\cosh l_{n-2})l'_{n-1} - (\cosh l_{n+2})l'_{n+1}.$$

It is convenient to subdivide Lemma 6.3 into two parts, Lemma 8.2 and Lemma 8.3, depending on whether or not the common perpendicular in question is adjacent to α . We begin with a somewhat more detailed discussion of the possible configurations.

Let P be a pair of pants in $S \setminus \mathcal{P}_{\mathcal{L}_t}$ that has α as a boundary component. For clarity, we distinguish between the three boundary curves α, β, γ of P and their projections $\pi(\alpha), \pi(\beta), \pi(\gamma)$ to S . We may always assume that $\pi(\gamma) \neq \pi(\alpha)$. There are then two possible configurations depending on whether or not $\pi(\beta) = \pi(\alpha)$. Figure 6(a) represents the case in which $\pi(\beta) = \pi(\alpha)$ and Figure 6(b), the case in which $\pi(\beta) \neq \pi(\alpha)$. In (b), we do not rule out the possibility that $\pi(\beta) = \pi(\gamma)$. This leads to a dichotomy in the formulae used in the proofs, but not in the final estimates. Let d be the length of the perpendicular between α and γ and let h_α be the length of the shortest perpendicular from α to itself.

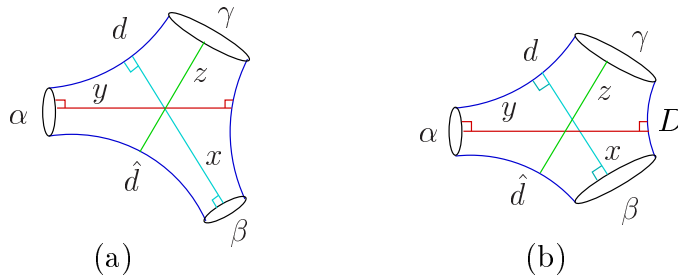


FIGURE 6. (a) $\pi(\alpha) = \pi(\beta)$; (b) $\pi(\alpha) \neq \pi(\beta)$.

Lemma 8.2. *Suppose that α is extremely short in \mathcal{L}_t . Then the derivatives of the perpendiculars adjacent to α are as follows:*

$$(i) \ d' = \frac{\partial d}{\partial l(\alpha)} \stackrel{*}{\asymp} -\frac{1}{l(\alpha)}, \quad (ii) \ h'_\alpha = \frac{\partial h_\alpha}{\partial l(\alpha)} \stackrel{*}{\asymp} -\frac{1}{l(\alpha)}.$$

Proof. Let x, y, z be the lengths of the perpendiculars as shown in Figure 6.

(i) In case (a), the formula (28) and the pentagon formula ([24] lemma 2.1) give, respectively,

$$d' \cosh x = -\frac{\cosh \hat{d}}{2}, \quad \cosh x = \sinh \hat{d} \sinh \frac{l(\alpha)}{2}.$$

If $l(\alpha)$ is small, then $\sinh l(\alpha) \stackrel{*}{\asymp} l(\alpha)$ and by Lemma 3.3, $\hat{d} \succ 1/\log l(\alpha)$ so that $\coth \hat{d} = O(1)$. Hence

$$d' = -\frac{\coth \hat{d}}{\sinh[l(\alpha)/2]} \stackrel{*}{\asymp} -\frac{1}{l(\alpha)}.$$

In case (b), using the same formulae as above, we get

$$d' \cosh x = \frac{1 - \cosh \hat{d}}{2}, \quad \cosh x = \sinh \hat{d} \sinh \frac{l(\alpha)}{2}.$$

Now by Lemma 3.3, $\hat{d} = 2 \log[1/l(\alpha)] \pm O(1)$ and therefore $\sinh \hat{d} \stackrel{*}{\asymp} 1/l(\alpha)^2 \stackrel{*}{\asymp} \cosh \hat{d}$. Hence,

$$d' = \frac{1 - \cosh \hat{d}}{2 \sinh \hat{d} \sinh[l(\alpha)/2]} \stackrel{*}{\asymp} l(\alpha) [1 - \cosh \hat{d}] \stackrel{*}{\asymp} -\frac{1}{l(\alpha)}.$$

(ii) In case (a), $h_\alpha = 2y$ and $\cosh y = \sinh d \sinh[l(\gamma)/2]$. Differentiating both sides, we get

$$y' = \frac{\sinh[l(\gamma)/2] \cosh d}{\sinh y} d'.$$

By Lemma 3.3, $\cosh d \stackrel{*}{\asymp} 1/[l(\alpha)l(\gamma)]$ and $\sinh y \stackrel{*}{\asymp} 1/l(\alpha)$, so we obtain $y' \stackrel{*}{\asymp} -1/l(\alpha)$, as desired.

In case (b), $h_\alpha = \hat{d}$ and by [24] equation(6),

$$\hat{d}' \cosh z = -\cosh d.$$

Substituting $\cosh z \stackrel{*}{\asymp} 1/l(\gamma)$ and $\cosh d \stackrel{*}{\asymp} 1/[l(\alpha)l(\gamma)]$ gives the desired estimate. \square

Now consider perpendiculars in P that are disjoint from α . Let h_γ be the length of the perpendicular from γ to itself, as shown in Figure 7. Further, if $\pi(\beta) \neq \pi(\alpha)$, let D be the length of the perpendicular between β, γ . We have $h_\gamma = D$ when $\pi(\beta) = \pi(\gamma)$ (see Figure 7(b)).

Lemma 8.3. *Suppose that α is extremely short in \mathcal{L}_t . Then the derivatives of the perpendiculars not adjacent to α are as follows:*

$$(i) \ h'_\gamma = \frac{\partial h_\gamma}{\partial l(\alpha)} \stackrel{*}{\asymp} l(\alpha), \quad (ii) \ D' = \frac{\partial D}{\partial l(\alpha)} \stackrel{*}{\asymp} l(\alpha).$$

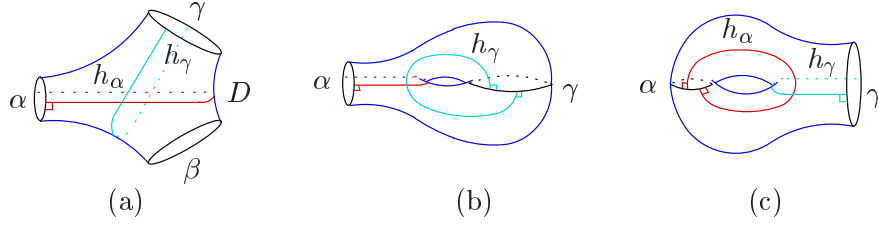


FIGURE 7. (a) $\pi(\alpha), \pi(\beta), \pi(\gamma)$ all distinct; (b) $\pi(\beta) = \pi(\gamma)$; (c) $\pi(\alpha) = \pi(\beta)$

Proof. (i) Assume that $\pi(\beta) \neq \pi(\gamma)$ so that we are in the configuration of Figure 7 (a) or (c). Consider the ‘front’ hexagon in P and denote the lengths of the sides as shown in Figure 8, so that $l_1 = l(\alpha)/2$, $l_2 = l(\gamma)/2$, and $z = h_\gamma/2$. By the pentagon formula,

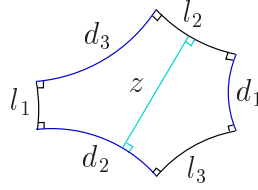


FIGURE 8. Front right-angled hexagon.

$\cosh z = \sinh d_3 \sinh l_1$. Taking the derivative with respect to l_1 , we get

$$z' \sinh z = d'_3 \cosh d_3 \sinh l_1 + \sinh d_3 \cosh l_1.$$

In the proof of Lemma 8.2, we had $d'_3 = -\coth d_2 / \sinh l_1$, and by the cosine formula for right-angled hexagons, we have

$$\cosh l_1 = \frac{\cosh d_1 + \cosh d_2 \cosh d_3}{\sinh d_2 \sinh d_3}.$$

Substituting these, we get

$$z' = \frac{1}{\sinh z} \cdot \frac{\cosh d_1}{\sinh d_2}.$$

Now by the sine formula for right-angled hexagons,

$$\frac{1}{\sinh d_2} = \frac{\sinh l_1}{\sinh d_1 \sinh l_2}.$$

With this, we have

$$z' = \sinh l_1 \cdot \frac{\cosh d_1}{\sinh d_1} \cdot \frac{1}{\sinh l_2 \sinh z} \stackrel{*}{\asymp} \sinh l_1 \stackrel{*}{\asymp} l_1,$$

since $\coth d_1 \stackrel{*}{\asymp} 1$ and $\sinh z \stackrel{*}{\asymp} 1/l_2$ when l_1, l_2 are respectively, bounded.

In the case $\pi(\beta) = \pi(\gamma)$, we have $h'_\gamma = D'$, which is computed below.
 (ii) Let y be the length of the perpendicular between α and the common perpendicular of β, γ , as in Figure 6. Then by Equation (28) and the pentagon formula, we have

$$D' \cosh y = 1, \quad \cosh y = \sinh d \sinh \frac{l(\gamma)}{2}.$$

By Lemma 3.3,

$$d = \log[1/l(\alpha)] + \log[1/l(\gamma)] \pm O(1)$$

and therefore $\sinh d \gtrsim 1/[l(\alpha)l(\gamma)]$. Since the pants system is short, $\sinh[l(\gamma)/2] \gtrsim l(\gamma)$. Thus $D' \gtrsim l(\alpha)$, as claimed. \square

Lemmas 8.2 and 8.3 together prove Lemma 6.3.

Proof of Lemma 7.2. If ξ has a component φ whose support is contained in Q , then $\varphi \cap Q = \varphi = \varphi_Q$, which has no effect on the inequality. Thus, we assume that no component of ξ has support entirely contained in Q . Then, for simplicity, let us further assume that ξ is a simple closed curve. Since both sides of the inequality scale linearly with weights, it is sufficient to prove the lemma under this assumption. The basic idea is to approximate an arc η of $\xi \cap Q$ with the union of η_Q and segments $p\hat{p}, q\hat{q}$ which run along ∂P between the endpoints \hat{p}, \hat{q} of η_Q and the corresponding endpoints p, q of η . Let α_p, α_q denote the components of ∂P that contain p, q , respectively. It is possible that $\alpha_p = \alpha_q$.

We will show that there are uniform constants C, C' such that

$$(29) \quad |l_\sigma(p\hat{p}) - l_\sigma(\alpha_p) \cdot Tw_\sigma(\xi, \alpha_p)/2| < C$$

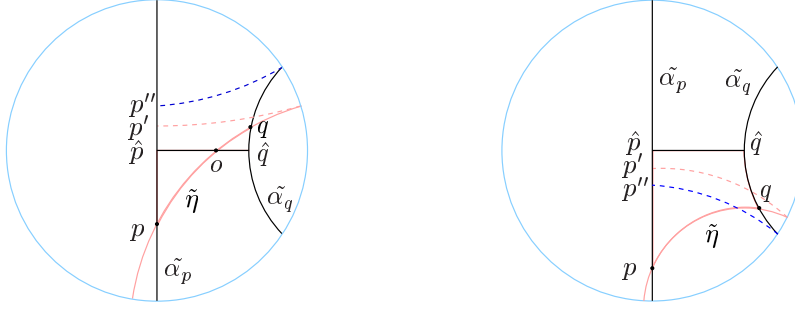
$$|l_\sigma(q\hat{q}) - l_\sigma(\alpha_q) \cdot Tw_\sigma(\xi, \alpha_q)/2| < C$$

$$(30) \quad |l_\sigma(\eta) - [l_\sigma(p\hat{p}) + l_\sigma(\eta_Q) + l_\sigma(q\hat{q})]| < C'.$$

It is convenient to consider the picture in the universal cover \mathbb{H}^2 , as shown in Figure 9. Let $\tilde{\eta}$ be a lift of η and let $\tilde{\alpha}_p, \tilde{\alpha}_q$ be the lifts of α_p, α_q that contain the endpoints of $\tilde{\eta}$. Since η_Q is homotopic to η relative to ∂Q and is perpendicular to ∂Q , its lift $\tilde{\eta}_Q$ is the unique perpendicular between $\tilde{\alpha}_p, \tilde{\alpha}_q$, drawn as the segment $\hat{p}\hat{q}$ in the figure. There are two possible cases, depending on whether or not $\tilde{\eta}$ intersects $\tilde{\eta}_Q$.

Let p', p'' be the feet of the perpendiculars as shown. That is, p' is the foot of the projection from the geodesic ray extending $\tilde{\eta}$ to $\tilde{\alpha}_p$ and p'' is the foot of the projection from $\tilde{\alpha}_q$ to $\tilde{\alpha}_p$. In either case, by definition of twist,

$$Tw_\sigma(\xi, \alpha_p, p) = 2l_\sigma(pp')/l_\sigma(\alpha_p)$$

FIGURE 9. Approximating the length of $\tilde{\eta}$.

and furthermore,

$$(31) \quad |l_\sigma(p\hat{p}) - l_\sigma(pp')| < l_\sigma(\hat{p}p'').$$

On the other hand, by trigonometry in \mathbb{H}^2 , we have

$$\cosh l_\sigma(\hat{p}p'') = 1 / \tanh l_\sigma(\hat{p}\hat{q}) = 1 / \tanh l_\sigma(\eta_Q).$$

Since η_Q goes from α_p to α_q and since by hypothesis $l_\sigma(\alpha_p), l_\sigma(\alpha_q) < \ell$, it follows from the collar lemma that $l_\sigma(\eta_Q) > c(\ell)$ for some constant $c(\ell)$ depending only on ℓ . This implies that there is a constant $C = C(\ell)$ depending only on ℓ such that $l_\sigma(\hat{p}p'') < C(\ell)$. Therefore, Equation (31) gives Equation (29), as desired. Of course, the same argument applies to α_q so

$$(32) \quad |l_\sigma(q\hat{q}) - l_\sigma(\alpha_q) \cdot Tw_\sigma(\xi, \alpha_q, q)/2| < C.$$

To show Equation (30), we use the well known fact that for any $\theta_0 > 0$, there exists a constant $k(\theta_0)$ such that for any hyperbolic triangle with sidelengths a, b, c and angle θ opposite to c with $\theta \geq \theta_0$, we have $a + b - c < k(\theta_0)$. In the case where $\tilde{\eta}$ intersects $\tilde{\eta}_Q$, as in the figure on the left, we apply this to the triangles $\triangle opp\hat{p}$ and $\triangle oq\hat{q}$ and get

$$l_\sigma(p\hat{p}) + l_\sigma(\hat{p}\hat{q}) + l_\sigma(q\hat{q}) - l_\sigma(pq) < k(\pi/2).$$

In the case on the right, we apply this to triangles $\triangle p\hat{q}\hat{p}$ and $\triangle pq\hat{q}$. To see that $\angle q\hat{q}p$ is bounded below by some θ_0 , observe that

$$\angle q\hat{q}p = \pi/2 - \angle p\hat{q}\hat{p}$$

and that

$$\sin(\angle p\hat{q}\hat{p}) < \frac{1}{\cosh l_\sigma(\hat{p}\hat{q})}.$$

Since $l_\sigma(\hat{p}\hat{q}) = l_\sigma(\tilde{\eta}_Q) > c(\ell)$, it follows that $\angle p\hat{q}\hat{p}$ is bounded away from $\pi/2$ and so $\angle q\hat{q}p$ is bounded below by some constant $\theta_0 = \theta_0(\ell)$,

as desired. Thus in this case,

$$l_\sigma(p\hat{p}) + l_\sigma(\hat{p}\hat{q}) + l_\sigma(q\hat{q}) - l_\sigma(pq) < k(\pi/2) + k(\theta_0(\ell)),$$

completing the proof of Equation (30).

Combining Equations (29), (30), and (32) we obtain

$$\left| l_\sigma(\eta) - \left[l_\sigma(\eta_Q) + l_\sigma(\alpha_q) \frac{Tw_\sigma(\xi, \alpha_q)}{2} + l_\sigma(\alpha_p) \frac{Tw_\sigma(\xi, \alpha_p)}{2} \right] \right| < K(\ell).$$

Summing over all arcs η in $\xi \cap Q$, we obtain

$$\left| l_\sigma(\xi \cap Q) - \left[l_\sigma(\xi_Q) + \sum_j l_\sigma(\alpha_j) \frac{Tw_\sigma(\xi, \alpha_j)}{2} i(\xi, \alpha_j) \right] \right| < K(\ell) i(\xi, \partial Q). \quad \square$$

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